1. (30 points) Identify each of the following series as absolutely convergent, conditionally convergent, or divergent. Justify your work and name any test that you use.

(a) \[ \sum_{n=2}^{\infty} \frac{1}{n \ln(n^2)} \]

(b) \[ \sum_{j=1}^{\infty} \frac{(3j)^j}{e^j(j+1)^j} \]

(c) \[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k}{k^2 + 4} \]

Solution:

(a) Divergent by the Integral Test. Let \( f(x) = \frac{1}{x \ln(x^2)} = \frac{1}{2x \ln(x)} \), a positive, continuous, decreasing function for \( x \geq 2 \).

\[
\int_{2}^{\infty} \frac{dx}{2x \ln(x)} = \int_{\ln 2}^{\infty} \frac{du}{2u} = \lim_{t \to \infty} \left[ \frac{1}{2} \ln |u| \right]_{\ln 2}^{t} = \lim_{t \to \infty} \frac{1}{2} (\ln t - \ln(\ln 2)) = \infty.
\]

Since the integral diverges, the given series also diverges.

(b) Divergent by Root Test,

\[
\lim_{n \to \infty} (|a_n|)^{1/n} = \lim_{n \to \infty} \left( \frac{(3n)^n}{e^n(n+1)^n} \right)^{1/n} = \lim_{n \to \infty} \frac{3n}{e(n+1)} = \frac{3}{e} > 1
\]

so the series is divergent by the Root Test.

Alternate Solution.

Divergent by the Ratio Test.

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(3(n+1))^{n+1}}{e^{n+1}(n+2)^{n+1}} \cdot \frac{e^n(n+1)^n}{(3n)^n} = \lim_{n \to \infty} \frac{1}{e} \cdot \frac{3(n+1)}{n+2} \left( \frac{3(n+1)^2}{3n(n+2)} \right)^n
\]

\[ \equiv \lim_{n \to \infty} \left| \frac{3}{e} \cdot \left( 1 + \frac{1}{n(n+2)} \right)^n \right| \leq \frac{3}{e} > 1 \]

so the series is divergent by the Ratio Test.
(c) Conditionally convergent since $\sum |a_n|$ is divergent by Limit Comparison Test with the harmonic series,

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n/(n^2 + 4)}{1/n} = \lim_{n \to \infty} \frac{n^2}{n^2 + 4} = \text{DOP} = 1$$

and note $0 < 1 < \infty$ implies that the series is divergent by Limit Comparison Test. Now, note that

$$\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} \frac{n}{n^2 + 4} = 1/n$$

thus $b_n = \frac{n}{n^2 + 4}$ is decreasing for $n > 2$ and, furthermore,

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{n}{n^2 + 4} = 0$$

thus we see that $\sum_{n>2} (-1)^{n-1} \frac{n}{n^2 + 4}$ is (conditionally) convergent by the Alternating Series Test.

**Alternate Solution.**

Instead of using the Limit Comparison Test, one can show that $\sum |a_n|$ is divergent by the Direct Comparison Test:

$$\frac{n}{n^2 + 4} = \frac{2n^2}{2n(n^2 + 4)} > \frac{n^2 + 4}{2n(n^2 + 4)} = \frac{1}{2n}$$

for $n > 2$ and

$$\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$$

is half of the divergent harmonic series so $\sum |a_n|$ diverges.

2. (15 Points) Suppose a power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ has an interval of convergence of $(2, 8)$. Use this information to answer the following questions. (No justification is necessary for your answers on this problem.)

(a) Find the center and radius of convergence.

(b) Is $\sum_{n=0}^{\infty} c_n 4^n$ absolutely convergent, conditionally convergent, or neither?

(c) For what values of $b$ does $\sum_{n=0}^{\infty} c_n b^n$ converge?

(d) If the Ratio Test is used to find the interval of convergence, what would $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ equal, where $a_n = c_n (x - a)^n$?

**Solution:**

(a) The interval $(2, 8)$ corresponds to $|x - a| < R$ where the center $a = 5$ and the radius $R = 3$.

(b) The value $x = 5 = 4$ lies outside the radius of convergence so the series diverges.

(c) Since $b = x - 5$, the series converges when $|b| < 3$.

(d) Let $L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Since $L < 1$ must match the interval of convergence $|x - 5| < 3$, the limit value $L = \frac{|x - 5|}{3}$.

3. (20 points) Let $f(x) = \left( 1 + \frac{x}{2} \right)^{1/3}$
(a) Find the first three nonzero terms of the Maclaurin series for $f$. Simplify your answer.

(b) What is the radius of convergence of the Maclaurin series for $f$? (Hint: You do not need to do a ratio test to answer this question.)

(c) Use part (b) to help determine if \( \sum_{i=1}^{\infty} \frac{1}{2^i} \left( \frac{1}{3} \right)^i \) converges or diverges. If it converges, determine its sum.

Solution:

(a) Using the Binomial Series formula with $k = \frac{1}{3}$ we have

\[
T_2(x) = \left( \frac{1}{3} \right)^0 \left( \frac{x}{2} \right)^0 + \left( \frac{1}{3} \right)^1 \left( \frac{x}{2} \right)^1 + \left( \frac{1}{3} \right)^2 \left( \frac{x}{2} \right)^2 = 1 + \frac{x}{6} - \frac{x^2}{36}
\]

which gives the first three nonzero terms of the Maclaurin series for $f$.

(b) We first observe that the binomial series for $f(x) = \left(1 + \frac{x}{2} \right)^{1/3}$ is given by

\[
f(x) = \sum_{i=0}^{\infty} \binom{1/3}{i} \frac{x^i}{2^i}
\]

and that the interval of convergence is given by $\left| \frac{x}{2} \right| < 1$ which implies that $-2 < x < 2$. Hence, the radius of convergence is $R = 2$.

(c) If we evaluate the binomial series

\[
f(x) = \left(1 + \frac{x}{2} \right)^{1/3} = \sum_{i=0}^{\infty} \binom{1/3}{i} \frac{x^i}{2^i}
\]

at $x = 1$ then we obtain

\[
f(1) = \sum_{i=0}^{\infty} \binom{1/3}{i} \frac{1}{2^i} \bigg|_{x=1} = \sum_{i=0}^{\infty} \frac{1}{2^i} \binom{1/3}{i} = \left(1 + \frac{x}{2} \right)^{1/3} \bigg|_{x=1} = \frac{3}{\sqrt[3]{2}}
\]

We know that a Taylor series converges absolutely for all $x$ values inside the radius of convergence. Since $x = 1$ is inside the interval of convergence, the given series converges absolutely. Finally, we note that the problem asked for the sum of $\sum_{i=1}^{\infty} \frac{1}{2^i} \binom{1/3}{i}$, which starts at $i = 1$. Hence,

\[
\sum_{i=1}^{\infty} \frac{1}{2^i} \binom{1/3}{i} = \sum_{i=0}^{\infty} \frac{1}{2^i} \binom{1/3}{i} - 1 = \frac{3}{\sqrt[3]{2}} - 1
\]
4. (15 points) Consider the series \(1 + x^2 + x^4 + x^6 + \cdots = \sum_{n=0}^{\infty} x^{2n}\).

(a) Find the sum of the series.

(b) Find the sum of the series \(\sum_{n=1}^{\infty} 2nx^{2n}\).

(c) Find the sum of the series \(\sum_{n=1}^{\infty} \frac{2n}{3^n}\).

Solution:

(a) This is a geometric series with first term \(a = 1\) and ratio \(r = x^2\). The sum of the series is \(\frac{a}{1-r} = \frac{1}{1-x^2}\).

(b) Use the derivative of the given series.

\[
\sum_{n=1}^{\infty} 2nx^{2n} = x \sum_{n=1}^{\infty} 2nx^{2n-1} = x \cdot \frac{d}{dx} \left( \sum_{n=0}^{\infty} x^{2n} \right) = x \cdot \frac{d}{dx} \left( \frac{1}{1-x^2} \right) = \frac{2x^2}{(1-x^2)^2}.
\]

(c) Use the result from part (b) and let \(x = \frac{1}{\sqrt{3}}\).

\[
\sum_{n=1}^{\infty} \frac{2n}{3^n} = \sum_{n=1}^{\infty} 2n \left( \frac{1}{\sqrt{3}} \right)^{2n} = \sum_{n=1}^{\infty} 2nx^{2n} \bigg|_{x=1/\sqrt{3}} = \frac{2x^2}{(1-x^2)^2} \bigg|_{x=1/\sqrt{3}} = \frac{2/3}{(1-1/3)^2} = \frac{2}{3}.
\]

5. (20 points) Suppose \(f(x)\) has a Taylor series representation, convergent for all \(x\), centered at \(a = 2\), \(f(2) = e^6\), and \(f^{(n)}(x) = (-1)^n 3^n e^{-3x}\) for \(n = 1, 2, 3, \ldots\).

(a) Write down the Taylor series (use sigma notation) and find the sum of the series.

(b) Find \(T_2(x)\), the 2nd order Taylor polynomial.

(c) Use Taylor’s formula to find an error bound if \(T_2(x)\) is used to approximate \(f(x)\) for \(2.5 < x < 3\).

Solution:

(a) \(f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n e^{-6}}{n!} (x-2)^n\).

Since \(f'(x) = -3e^{-3x}\) and \(f(2) = e^{-6}\), the sum of the series is \(f(x) = e^{-3x}\).

(b) \(T_2(x) = e^{-6} - 3e^{-6}(x-2) + \frac{9}{2} e^{-6}(x-2)^2\).

(c) The error term is \(R_2(x) = \frac{f'''(z)}{3!} (x-2)^3\). Since \(2.5 < x < 3\), the term \(|(x-2)^3|\) has a maximum value when \(x = 3\). Since \(2 < z < 3\), the derivative \(|f'''(z)| = 3^3 e^{-3z}\) has a maximum value when \(z = 2\). It follows that

\[
|R_2(x)| = \left| \frac{f'''(z)}{3!} (x-2)^3 \right| < \frac{3^3 e^{-3(2)}}{3!} (3-2)^3 = \frac{9}{2} e^{-6}.
\]
Frequently Used Maclaurin Series

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad R = 1
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R = \infty
\]

\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad R = \infty
\]

\[
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad R = \infty
\]

\[
\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \quad R = 1
\]

\[
\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad R = 1
\]

\[
(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad R = 1
\]