1. (36 points) Evaluate the following integrals. Simplify your answers.

(a) \[\int_7^{\sqrt{2}} \frac{\sqrt{t^2 - 49}}{t} \, dt\]

(b) \[\int_1^4 x^{1/2} \ln(x) \, dx\]

(c) \[\int \frac{\sin(x)}{\cos^2(x) - 3 \cos(x)} \, dx\]

Solution:

(a) Trig substitution. Let \(t = 7 \sec \theta\). Then \(dt = 7 \sec \theta \tan \theta \, d\theta\) and \(\sqrt{t^2 - 49} = \sqrt{49 \sec^2 \theta - 49} = \sqrt{49 \tan^2 \theta} = 7 \tan \theta\). We can convert the limits \(x = 7\) and \(x = 7\sqrt{2}\) to \(\theta = 0\) and \(\theta = \pi/4\), respectively.

\[
\int_7^{\sqrt{2}} \frac{\sqrt{t^2 - 49}}{t} \, dt = \int_0^{\pi/4} \frac{7 \tan \theta}{7 \sec \theta} (7 \sec \theta \tan \theta) \, d\theta = \int_0^{\pi/4} 7 \tan^2 \theta \, d\theta \\
= \int_0^{\pi/4} 7 (\sec^2 \theta - 1) \, d\theta = \left[7(tan \theta - \theta)\right]_0^{\pi/4} \\
= 7 \left(1 - \frac{\pi}{4}\right)
\]

(b) Integration by parts: Let \(u = \ln x\), \(du = (1/x) \, dx\), \(dv = x^{1/2} \, dx\) and \(v = (2/3)x^{3/2}\). Then:

\[
\int_1^4 x^{1/2} \ln(x) \, dx = \left[\frac{2}{3}x^{3/2} \ln x\right]_1^4 + \int_1^4 \frac{2}{3}(x^{3/2}) \, dx \\
= \frac{2}{3}(4)^{3/2} \ln 4 - \left[\frac{2}{3} \cdot \frac{2}{3} x^{3/2}\right]_1^4 \\
= \frac{16}{3} \ln 4 - \frac{4}{9} (4^{3/2} - 1) \\
= \frac{16}{3} \ln 4 - \frac{28}{9}
\]
(c) Let \( u = \cos(x) \) then we have
\[
\int \frac{\sin(x) \, dx}{\cos^2(x) - 3 \cos(x)} = \int \frac{-1}{u(u - 3)} \, du = \frac{1}{3} \int \frac{1}{u} \, du - \frac{1}{3} \int \frac{1}{u - 3} \, du = \frac{1}{3} \ln |u| - \frac{1}{3} \ln |u - 3| + C
\]
so,
\[
\int \frac{\sin(x) \, dx}{\cos^2(x) - 3 \cos(x)} = \frac{1}{3} \ln \left| \frac{\cos(x)}{\cos(x) - 3} \right| + C
\]

2. (14 points) For this problem, let 
\[ I = \int_0^1 \frac{1}{1 + x^2} \, dx. \]

(a) Calculate the value of \( I \).

(b) Now, estimate \( I \) using the trapezoidal approximation \( T_2 \).

(c) In this problem, \( f(x) = \frac{1}{1 + x^2} \), \( f''(x) = \frac{2(3x^2 - 1)}{(1 + x^2)^3} \) and \( f^{(3)}(x) > 0 \) on \((0, 1)\). Use this information to find an error estimate for \( T_2 \). Briefly explain your reasoning.

(d) Express \( \pi \) in terms of \( I \) using your answer from part (a). Find an approximate value of \( \pi \) using your answer from part (b). Simplify your answer. (Remark: Note that we could use this method to estimate the value of \( \pi \) numerically to greater precision by using larger values of \( n \).)

Solution:

(a) \( I = \int_0^1 \frac{1}{1 + x^2} \, dx = \left[ \arctan(x) \right]_0^1 = \frac{\pi}{4} \)

(b) If \( n = 2 \) subintervals and \( \Delta x = 1/2 \), then
\[
T_2 = \frac{\Delta x}{2} \left( f(0) + 2f \left( \frac{1}{2} \right) + f(1) \right) = \frac{1}{2} \cdot \frac{1}{2} \left( 1 + 2 \cdot \frac{4}{5} + 1 \right) = \frac{31}{40}
\]

(c) Since \( f^{(3)}(x) > 0 \), \( f'' \) is an increasing function. At the left endpoint \( f''(0) = -2 \) and at the right endpoint \( f''(1) = 1/2 \). Let \( K \) equal the maximum value of \( |f''| = 2 \).
\[
E_T \leq K(b - a)^3 = \frac{2(1)^3}{12(2)^2} = \frac{1}{24}
\]

(d) In part (a) we found that \( I = \pi/4 \) so \( \pi = 4I \) and an approximate value of \( \pi \) is \( 4 \times \frac{31}{40} = 3.1 \).

3. (16 points) Determine whether the following integrals are convergent or divergent. Explain your reasoning.

(a) \( \int_0^\infty \frac{1}{x^2} \, dx \)

(b) \( \int_1^\infty \frac{x^2}{x^5 + 2} \, dx \)

Solution:

(a) For \( \int_0^\infty \frac{1}{x^2} \, dx \) to converge, both of the integrals \( \lim_{a \to 0^+} \int_a^1 \frac{1}{x^2} \, dx \) and \( \lim_{t \to \infty} \int_1^t \frac{1}{x^2} \, dx \) must converge. We see that:
\[
\lim_{a \to 0^+} \int_a^1 \frac{1}{x^2} \, dx = \lim_{a \to 0^+} \left[ -\frac{1}{x} \right]_a^1 = \lim_{a \to 0^+} \left[ -1 + \frac{1}{a} \right]
\]
which diverges. Therefore, the original integral, \( \int_0^\infty \frac{1}{x^2} \, dx \) diverges. (Note, the second integral, \( \lim_{t \to \infty} \int_1^t \frac{1}{x^2} \, dx \) does converge. But, once we found that the first integral diverged, we didn’t need to look at the second integral.)
(b) We note that for \( x \geq 1 \), we have \( 0 \leq \frac{x^2}{x^5 + 2} \leq \frac{x^2}{x^3} = \frac{1}{x^3} \). Also,

\[
\int_1^\infty \frac{1}{x^3} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^3} \, dx = \lim_{t \to \infty} \left[ \frac{-1}{2x^2} \right]_1^t = \lim_{t \to \infty} \left[ \frac{-1}{2t^2} + \frac{1}{2} \right] = \frac{1}{2}
\]

Therefore, by the comparison test, the original integral converges.

4. (20 points) Four unrelated, short answer questions. For the True/False questions, if the statement is true, write the word TRUE and give a short explanation of how you know it is true. If the statement is false, write the word FALSE and give a counterexample that shows the statement is false.

(a) If \( f(x) \leq g(x) \) and \( \int_1^\infty g(x) \, dx \) converges then \( \int_1^\infty f(x) \, dx \) also converges. True or False? Explain.

(b) If \( f \) is a continuous, decreasing function on \( [1, \infty) \), and if \( \lim_{x \to \infty} f(x) = 0 \) then \( \int_1^\infty f(x) \, dx \) is convergent. True or False? Explain.

(c) If \( x = 3 \sin \theta \) is used as a trigonometric substitution for \( \int f(x) \, dx \) and the result is \( \int f(x) \, dx = -\cot \theta - \theta + C \), what is the final answer in terms of \( x \)? Simplify your answer.

(d) Give the form of the partial fraction decomposition of the rational function \( \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \). Do not find the values of the coefficients.

Solution:

(a) False. For example, suppose \( f(x) = -1/x \) and \( g(x) = 1/x^2 \). Then, \( f(x) = -1/x \leq 1/x^2 = g(x) \) for all \( x \geq 1 \). And, \( \int_1^\infty g(x) \, dx \) converges but \( \int_1^\infty f(x) \, dx \) diverges.

(b) False. Consider the function \( f(x) = 1/x \) with domain \( [1, \infty) \). On this domain, \( f(x) = 1/x \) is continuous and decreasing and \( \lim_{x \to \infty} f(x) = 0 \). However, \( \int_1^\infty 1/x \, dx \) diverges.

(c) Since \( \sin \theta = x/3 \), you can set up a reference triangle with the opposite side having length \( x \) and the hypotenuse length 3. Then, the adjacent side is \( \sqrt{9 - x^2} \), so we have \( \cot \theta = \frac{\sqrt{9 - x^2}}{x} \) and we obtain:

\[
\int f(x) \, dx = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C
\]

(d) First note that \( 2x^3 + 3x^2 - 2x = x(2x - 1)(x + 2) \). Then,

\[
\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}
\]

5. (14 points) Find the area of the shaded region between the graphs of \( f(x) = \sin^2(x) \) and \( g(x) = \cos^2(x) \) in the figure below.
Solution: Find the points of graph intersection:

\[
\sin^2(x) = \cos^2(x) \\
\cos^2(x) - \sin^2(x) = 0 \\
\cos(2x) = 0 \\
2x = \frac{\pi}{2}, \frac{3\pi}{2} \\
x = \frac{\pi}{4}, \frac{3\pi}{4}
\]

We have \( \sin^2(x) \geq \cos^2(x) \) for \( \pi/4 \leq x \leq 3\pi/4 \). Then,

\[
\text{Area} = \int_{\pi/4}^{3\pi/4} (\sin^2(x) - \cos^2(x)) \, dx \\
= -\int_{\pi/4}^{3\pi/4} \cos(2x) \, dx \\
= -\left[ \frac{1}{2} \sin(2x) \right]_{\pi/4}^{3\pi/4} \\
= -\frac{1}{2} \left[ \sin \left( \frac{3\pi}{2} \right) - \sin \left( \frac{\pi}{2} \right) \right] \\
= -\frac{1}{2} [-1 - 1] = 1
\]

Some Trigonometric identities

\( 2 \cos^2(x) = 1 + \cos(2x) \)
\( 2 \sin^2(x) = 1 - \cos(2x) \)
\( \sin(2x) = 2 \sin(x) \cos(x) \)
\( \cos(2x) = \cos^2(x) - \sin^2(x) \)

Inverse Trigonometric Integral Identities

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C, \quad u^2 < a^2 \\
\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}(u/a) + C \\
\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}|u/a| + C, \quad u^2 > a^2
\]

Midpoint Rule

\[
\int_a^b f(x) \, dx \approx \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad \text{where} \quad \Delta x = \frac{b - a}{n} \quad \text{and} \quad \bar{x}_i = \frac{x_{i-1} + x_i}{2}
\]
\[ |E_M| \leq \frac{K(b - a)^3}{24n^2}. \]

**Trapezoidal Rule**

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \quad \text{where } \Delta x = \frac{b - a}{n}
\]

and \(|E_T| \leq \frac{K(b - a)^3}{12n^2}\).