On the front of your bluebook, please write: a grading key, your name, student ID, your lecture number and instructor. This exam is worth 100 points and has 5 questions on both sides of this paper.

- Include this exam sheet in your bluebook. However, nothing on this exam sheet will be graded. Make sure all of your work is in your bluebook.
- **Show all work!** Answers with no justification will receive no points.
- Please begin each problem on a new page.
- No notes or papers, calculators, cell phones, or electronic devices are permitted.

1. (36 points) Evaluate the following integrals. Simplify your answers.
   (a) \[ \int_{7}^{\sqrt{2}} \frac{\sqrt{t^2 - 49}}{t} \, dt \]
   (b) \[ \int_{1}^{4} x^{1/2} \ln(x) \, dx \]
   (c) \[ \int \frac{\sin(x)}{\cos^2(x) - 3 \cos(x)} \, dx \]

2. (14 points) For this problem, let \( I = \int_{0}^{1} \frac{1}{1 + x^2} \, dx \).
   (a) Calculate the value of \( I \).
   (b) Now, estimate \( I \) using the trapezoidal approximation \( T_2 \).
   (c) In this problem, \( f(x) = \frac{1}{1 + x^2} \), \( f''(x) = \frac{2(3x^2 - 1)}{(1 + x^2)^3} \) and \( f^{(3)}(x) > 0 \) on \((0, 1)\). Use this information to find an error estimate for \( T_2 \). Briefly explain your reasoning.
   (d) Express \( \pi \) in terms of \( I \) using your answer from part (a). Find an approximate value of \( \pi \) using your answer from part (b). Simplify your answer. (Remark: Note that we could use this method to estimate the value of \( \pi \) numerically to greater precision by using larger values of \( n \).)

3. (16 points) Determine whether the following integrals are convergent or divergent. Explain your reasoning.
   (a) \[ \int_{0}^{\infty} \frac{1}{x^2} \, dx \]
   (b) \[ \int_{1}^{\infty} \frac{x^2}{x^5 + 2} \, dx \]

4. (20 points) Four unrelated, short answer questions. For the True/False questions, if the statement is true, write the word TRUE and give a short explanation of how you know it is true. If the statement is false, write the word FALSE and give a counterexample that shows the statement is false.
   (a) If \( f(x) \leq g(x) \) and \( \int_{1}^{\infty} g(x) \, dx \) converges then \( \int_{1}^{\infty} f(x) \, dx \) also converges. True or False? Explain.
   (b) If \( f \) is a continuous, decreasing function on \([1, \infty)\), and if \( \lim_{x \to \infty} f(x) = 0 \) then \( \int_{1}^{\infty} f(x) \, dx \) is convergent. True or False? Explain.
   (c) If \( x = 3 \sin \theta \) is used as a trigonometric substitution for \( \int f(x) \, dx \) and the result is \( \int f(x) \, dx = -\cot \theta - \theta + C \), what is the final answer in terms of \( x \)? Simplify your answer.

TURN OVER – MORE ON THE REVERSE
(d) Give the form of the partial fraction decomposition of the rational function \( \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \). Do not find the values of the coefficients.

5. (14 points) Find the area of the shaded region between the graphs of \( f(x) = \sin^2(x) \) and \( g(x) = \cos^2(x) \) in the figure below.

![Graph of \( y = \sin^2(x) \) and \( y = \cos^2(x) \)]

Some Trigonometric identities

\[
2 \cos^2(x) = 1 + \cos(2x) \\
2 \sin^2(x) = 1 - \cos(2x) \\
\sin(2x) = 2 \sin(x) \cos(x) \\
\cos(2x) = \cos^2(x) - \sin^2(x)
\]

Inverse Trigonometric integral identities

\[
\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C, \quad u^2 < a^2 \\
\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C \\
\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}|u/a| + C, \quad u^2 > a^2
\]

Midpoint Rule

\[
\int_a^b f(x) \, dx \approx \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad \bar{x}_i = \frac{x_{i-1} + x_i}{2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}.
\]

Trapezoidal Rule

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2}.
\]