1. For this problem let the region $\mathcal{R}$ be the region enclosed by the curve $y = \ln(-x)$ and the lines $x = 0$, $y = -1$, and $y = -4$.

(a) (6 pts) Find the area of the region $\mathcal{R}$.

(b) (6 pts) Suppose the region $\mathcal{R}$ is revolved about the line $y = -5$. Find the volume of the resulting solid using the Shell Method.

(c) (6 pts) Suppose we find a thin piece of metal in the shape of region $\mathcal{R}$ with density $\rho = 1.0690 \text{ kg/m}^2$, find $\bar{x}$, the $x$-coordinate of the center of mass of the thin piece of metal.

(d) (6 pts) Suppose the region $\mathcal{R}$ is revolved about the line $x = -2$. Set up, but do not solve, an integral (or integrals) to find the volume of the resulting solid using the Disk/Washer Method.

(e) (6 pts) Suppose a surface is generated by rotating the curve $y = \ln(-x)$, $-4 \leq y \leq -1$ about the line $y = 0$. Set up, but do not solve, an integral (or integrals) to find the surface area of this surface.

Solution:

(a) $A = \int_{-4}^{-1} e^y dy = e^y \bigg|_{-4}^{-1} = e^{-1} - e^{-4} = \frac{e^3 - 1}{e^4}$

(b) Here $\Delta V = 2\pi rh\Delta y$ where $r = y + 5$ and $h = e^y$, now we use integration by parts to evaluate the integral with $u = y + 5 \Rightarrow du = dy$ and $dv = e^y \Rightarrow v = e^y$, thus

$$V = \int_{-4}^{-1} 2\pi(y + 5)e^y dy = 2\pi \left[ e^y(y + 5) \bigg|_{-4}^{-1} - \int_{-4}^{-1} e^y dy \right] = 6\pi e^{-1}$$

(c) Here we have

$$\bar{x} = \frac{M_y}{M} = \frac{\int_{-4}^{-1} -\frac{e^y}{2} \rho e^y dy}{\int_{-4}^{-1} \rho e^y dy} = -\frac{1}{2} \int_{-4}^{-1} e^{2y} dy$$

and note that

$$\frac{1}{2} \int_{-4}^{-1} e^{2y} dy = -\frac{e^{2y}}{4} \bigg|_{-4}^{-1} = -\frac{e^{-2}}{4} + \frac{e^{-8}}{4} = \frac{1 - e^6}{4e^8}$$

and combining this with the result from part (a), we have

$$\bar{x} = \frac{1 - e^6}{4e^8} \cdot \frac{e^4}{e^3 - 1} = \frac{(1 + e^3)}{4e^4}.$$
(d) Here \( \Delta V = \pi [R^2 - r^2] \Delta y \) where \( R = 2 \) and \( r = 2 - e^y \) and so \( V = \int_{-4}^{-1} \pi [(2)^2 - (2 - e^y)^2] \ dy \).

(e) Here \( SA = \int 2\pi rdL \) where \( r = -y \) and \( dL = \sqrt{1 + g'(y)^2} \ dy \) where \( g(y) = e^y \) and so
\[
SA = \int 2\pi (-y) \sqrt{1 + g'(y)^2} \ dy = \int_{-4}^{-1} 2\pi (-y) \sqrt{1 + e^{2y}} \ dy = \int_{-1}^{-4} 2\pi y \sqrt{1 + e^{2y}} \ dy
\]
or, with respect to \( x \), we have \( r = -f(x) = -\ln(-x) \) and \( dL = \sqrt{1 + f'(x)^2} \ dx \) where \( f(x) = \ln(-x) \) and so
\[
SA = \int -e^{-4} 2\pi (-f(x)) \sqrt{1 + f'(x)^2} \ dx = \int -e^{-4} 2\pi (-\ln(-x)) \sqrt{1 + \frac{1}{x^2}} \ dx = \int -e^{-4} 2\pi \ln(-x) \sqrt{1 + \frac{1}{x^2}} \ dx
\]

2. The following problems are not related:

(a) (6 pts) Evaluate the integral \( \int_0^1 x^3 \sqrt{1 - x^2} \ dx \)

(b) (6 pts) What error estimate do we have if we approximate the integral \( \int_6^{10} f(x) \ dx \) using a Trapezoidal rule with 8 subintervals of equal length and if we know that \( -3 \leq f''(x) \leq 2 \) for all \( x \)?

(c) (6 pts) Determine if the sequence converges or diverges: \( a_n = \frac{1}{n} \int_1^n 1 \ dx \), where \( n = 1, 2, 3, \ldots \)

(d) (6 pts) Suppose we approximate \( f(x) = \sqrt{x} \) by \( T_1(x) \), a Taylor polynomial of order 1 centered at \( a = 2 \), use Taylor’s Formula to find an error bound of the approximation \( f(x) \approx T_1(x) \) if \( 4 \leq x \leq 4.2 \).

**Solution:**

(a) Using the trigonometric substitution \( x = \sin(\theta) \), we get
\[
\int_0^1 x^3 \sqrt{1 - x^2} \ dx = \int_0^{\pi/2} \sin^3(\theta) \cos^2(\theta) \ d\theta
\]
now note that \( \sin^3(\theta) = \sin(\theta) \sin(\theta) = [1 - \cos^2(\theta)] \sin(\theta) \) and so
\[
\int_0^{\pi/2} \sin^3(\theta) \cos^2(\theta) \ d\theta = \int_0^{\pi/2} [1 - \cos^2(\theta)] \cos^2(\theta) \sin(\theta) \ d\theta = \int_0^{\pi/2} [\cos^2(\theta) - \cos^4(\theta)] \sin(\theta) \ d\theta
\]
now let \( u = \cos(\theta) \) then \( du = -\sin(\theta) \ d\theta \) and we get
\[
\int_0^{\pi/2} [\cos^2(\theta) - \cos^4(\theta)] \sin(\theta) \ d\theta = -\int_1^0 (u^2 - u^4) \ du = \left[ \frac{u^3}{3} - \frac{u^5}{5} \right]_1^0 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}
\]

(b) Note that here \( b - a = 10 - 6 = 4 \), \( n = 8 \), and \( |f''(x)| \leq 3 = K \), so \( E_T \leq \frac{K(b-a)^3}{12n^2} = \frac{3 \cdot 4^3}{12 \cdot 8^2} = \frac{1}{4} \).

(c) Note that \( \int_1^n 1 \ dx = \ln |x| \bigg|_1^n = \ln(n) \), and so
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\ln(n)}{n} L'H \]
thus the sequence converges to 0.

(d) Note that the error term here is \( R_1(x) = \frac{f''(z)}{2!} (x - 2)^2 \) for some \( z \) between \( x \) and \( a = 2 \), also note that \( f''(x) = \frac{1}{4} x^{-3/2} \), now since \( z \) is between \( x \) and \( a = 2 \) and since \( 4 \leq x \leq 4.2 \) we have that \( 2 < z < 4 \) and
so \(\frac{1}{z} < \frac{1}{2}\) and thus \(z^{-3/2} = \left(\frac{1}{z}\right)^{3/2} < \left(\frac{1}{2}\right)^{3/2} = \frac{1}{\sqrt{8}}\) and also note \(2 \leq x - 2 \leq 2.2\) thus \(|x - 2|^2 < (2.2)^2\) and so the resulting error bound according to Taylor’s Formula is

\[
|\text{error}| = |R_1(x)| = \left| \frac{f''(z)}{2!} (x-2)^2 \right| = \frac{1}{2!} \cdot \frac{1}{4} \cdot \left(\frac{1}{z}\right)^{3/2} \cdot |x-2|^2 = \frac{1}{4} \cdot \frac{1}{\sqrt{8}} \cdot (2.2)^2 = \frac{(2.2)^2}{8\sqrt{8}} = \frac{(1.1)^2}{4\sqrt{2}}
\]

3. (a) Determine if the given series are absolutely convergent, conditionally convergent or divergent, find the sum of the series when possible, justify all answers:

\[
(i) \sum_{m=106}^{\infty} (-1)^{m-1} \frac{m+2}{m^2+4} \quad \text{(ii) } \sum_{m=106}^{\infty} \frac{m}{m^2+4}
\]

(b) Evaluate the integrals, show all work: (i) \(\int_0^1 \frac{x^2-x+1}{x^3+x} \, dx\) \(\text{ (ii) } \int \tan^{-1}(x) \, dx\)

Solution: (a) (i) Note that \(\sum_{m=106}^{\infty} \left| (-1)^{m-1} \frac{m}{m^2+4} \right| = \sum_{m=106}^{\infty} \frac{m}{m^2+4}\) and now we perform a Limit Comparison Test with the divergent harmonic series \(\sum_{m=1}^{\infty} \frac{1}{m}\), so we have

\[
\lim_{m \to \infty} \frac{m/(m^2+4)}{1/m} = \lim_{m \to \infty} \frac{m}{m^2+4} \cdot \frac{1}{m} = 1
\]

and since \(0 < 1 < \infty\), the series \(\sum_{m=106}^{\infty} \frac{m}{m^2+4}\) is divergent by the Limit Comparison Test. Now note that using the Alternating Series Test with \(b_m = \frac{m}{m^2+4}\), we have that

\[
b'_m = \frac{m^2+4-2m^2}{(m^2+4)^2} = \frac{4-m^2}{(m^2+4)^2} < 0 \text{ for } m > 2
\]

and \(\lim_{m \to \infty} \frac{m}{m^2+4} \stackrel{L'H}{\to} 0\) and so \(b_m\) is decreasing and goes to 0 and so \(\sum_{m=106}^{\infty} (-1)^{m-1} \frac{m}{m^2+4}\) is conditionally convergent by the Alternating Series Test and the work done previously.

(a) (ii) Note that,

\[
\frac{1}{2} + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{26} + \cdots = \left(\frac{1}{2} + 1\right) + \frac{1}{4} \left(\frac{1}{2} + 1\right) + \frac{1}{4^2} \left(\frac{1}{2} + 1\right) + \cdots
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{4^n} \left(\frac{3}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{4^n} \frac{3}{2} = \frac{3/2}{1-1/4} = 2
\]

so this is a convergent Geometric Series (since \(|r| = 1/4 < 1\)) which converges absolutely to 2.

(b) (i) Note that \(x^3 + x = x(x^2 + 1)\) and using partial fractions we have

\[
\frac{x^2-x+1}{x^2+1} = A + \frac{Bx+C}{x^2+1} \implies x^2-x+1 = (A+B)x^2 + Cx + A \implies A = 1, \quad B = 0, \quad C = -1
\]

thus

\[
\int \frac{x^2-x+1}{x^3+x} \, dx = \int \frac{dx}{x} - \int \frac{dx}{x^2+1} = \ln |x| - \tan^{-1}(x) + C
\]

and so

\[
\int_0^{\infty} \frac{x^2-x+1}{x^3+x} \, dx = \lim_{t \to 0^+} \int_t^{\infty} \frac{x^2-x+1}{x^3+x} \, dx = \lim_{t \to 0^+} \left[ \ln |x| - \tan^{-1}(x) \right]_{t}^{1} = \lim_{t \to 0^+} \left[ -\frac{\pi}{4} - \ln |t| - \tan^{-1}(t) \right] = -\infty
\]
thus the integral \( \int_0^1 \frac{x^2 - x + 1}{x^3 + x} \, dx \) diverges to \(-\infty\).

(b) (ii) Using integration by parts with \( u = \tan^{-1}(x) \Rightarrow du = \frac{1}{1 + x^2} \) and \( dv = dx \Rightarrow v = x \) yields

\[
\int \tan^{-1}(x) \, dx = x \tan^{-1}(x) - \frac{1}{2} \ln(1 + x^2) + C
\]

where in the last equality we used the \( u \)-substitution \( u = 1 + x^2 \).

---

4. Suppose you are told that the Maclaurin Series of \( \ln(1 + x) \) is \( \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \), and suppose you are given the function \( F(x) = \int_0^x \frac{\ln(1 + t)}{t} \, dt \) for \( 0 \leq x \leq 1 \).

(a) (6 pts) Find the Maclaurin series for the function \( F'(x) \). What is the interval of convergence of this Maclaurin series?

(b) (6 pts) Find the fifth-degree Taylor polynomial, \( T_5(x) \), of \( F(x) \) at 0.

(c) (6 pts) Estimate the accuracy of the polynomial in (b) as an approximation to the definite integral \( F(1) \).

**Solution:**

(a) \( F'(x) = \frac{\ln(1 + x)}{x} = x^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n-1}}{n} \) now note that

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^n}{n+1} \cdot \frac{n}{x^{n-1}} \right| = |x| \lim_{n \to \infty} \frac{n}{n+1} = |x|
\]

and so \( |x| \leq 1 \) implies \(-1 \leq x \leq 1\), now to check the endpoints. If \( x = -1 \) then we have

\[
F'(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]

which is a divergent \( p \)-series and if \( x = 1 \) then we have

\[
F'(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}
\]

which is convergent by the Alternating Series Test (here \( b_n = 1/n \) is decreasing and \( \lim_{n \to \infty} b_n = 0 \), so the interval of convergence is \([-1 < x \leq 1]\). (Note the convergence at \( x = 1 \) is conditional but the problem does not ask this.)

(b) Note that

\[
F(x) = \int_0^x \frac{\ln(1 + t)}{t} \, dt = \int_0^x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^{n-1}}{n} \, dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{n^2}\bigg|_0^x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n^2}
\]

thus \( F(x) = x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25} - \frac{x^6}{36} + \ldots \) and so \( T_5(x) = x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \frac{x^5}{25} \)

(c) From part (b) note that

\[
F(1) = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \ldots
\]

\[
T_5(1) = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots
\]

and so by the Alternating Series Error Estimation Theorem \( |\text{error}| = |F(1) - T_5(1)| \leq \frac{1}{36} \).
5. The following problems are not related:

(a) (9 pts.) Sketch the curve \( x = 4 \sin(\theta), y = \cos(\theta) \) for \( 0 \leq \theta \leq \pi \). Be sure to indicate the direction in which the curve is traced.

(b) (9 pts.) Find a Cartesian equation of the parametric curve \( x = \frac{1}{2} \cosh(t), y = 2 \sinh(t) \). Identify the type of Cartesian curve you have found.

(c) (6 pts.) Find \( \frac{dy}{dx} \) of the curve \( x(s) = 1 + 4s - s^2, y(s) = 2 - s^3 \) when \( s = 1 \).

(d) (6 pts) Find the exact length of the curve \( x = e^t + e^{-t}, y = 5 - 2t, 0 \leq t \leq 3 \).

Solution:

(a) Note that \( \sin^2(\theta) + \cos^2(\theta) = 1 \) implies \( \left(\frac{x}{4}\right)^2 + y^2 = 1 \) and so we have an ellipse, and note that when \( \theta = 0 \) we have \((x, y) = (0, 1)\) and when \( \theta = \pi \) we have \((x, y) = (0, -1)\), thus,

(b) Note that \( \cosh(t) = 2x \) and \( \sinh(t) = y/2 \) and from the formula sheet we know \( \cosh^2(t) - \sinh^2(t) = 1 \) so we have

\[
(2x)^2 - \left(\frac{y}{2}\right)^2 = 1 \implies \frac{x^2}{1/4} - \frac{y^2}{4} = 1 \text{ or } x = \frac{1}{2} \sqrt{1 + \frac{y^2}{4}}
\]

which yields a [hyperbola](note technically we only have the part of the hyperbola above the \( x \)-axis since \( x = \frac{1}{2} \cosh(t) > 0 \) for all \( t \)).

(c) Here we have, \( \left. \frac{dy}{dx} \right|_{s=1} = \left. \frac{dy}{ds} \right|_{dx/ds} = \frac{-3s^2}{4 - 2s} \bigg|_{s=1} = \frac{3}{2} \).

(d) Note that \( x'(t) = e^t - e^{-t} \) and \( y'(t) = -2 \) and so

\[
L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} \, dt = \int_0^3 \sqrt{(e^{2t} - 2 + e^{-2t}) + 4} \, dt
\]

\[
= \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}} \, dt = \int_0^3 \sqrt{(e^t + e^{-t})^2} \, dt = \int_0^3 (e^t + e^{-t}) \, dt
\]

thus \( L = \int_0^3 (e^t + e^{-t}) \, dt = (e^t - e^{-t}) \bigg|_0^3 = e^3 - e^{-3} = \frac{e^6 - 1}{e^3} \).

6. The following problems are not related:

(a) (9 pts.) Find the area enclosed by the curve \( r = 1 - \sin(\theta) \).

(b) (9 pts.) Find the area inside \( r = 1 - \sin(\theta) \) and outside \( r = 1 \).
(c) (6 pts.) Find the area of one loop of $r^2 = \sin(2\theta)$.

**Solution:**

(a) Using symmetry, we have

\[
A = 2 \left[ \frac{1}{2} \int_{\pi/2}^{3\pi/2} (1 - \sin(\theta))^2 \, d\theta \right] = \int_{\pi/2}^{3\pi/2} (1 - 2 \sin(\theta) + \sin^2(\theta)) \, d\theta
\]

\[
= \int_{\pi/2}^{\pi/2} \left( 1 - 2 \sin(\theta) + \frac{1}{2}(1 - \cos(2\theta)) \right) \, d\theta = \int_{\pi/2}^{3\pi/2} \left( \frac{3}{2} - 2 \sin(\theta) - \frac{1}{2} \cos(2\theta) \right) \, d\theta
\]

\[
= \left[ \frac{3}{2} \theta + 2 \cos(\theta) - \frac{\sin(2\theta)}{4} \right]_{\pi/2}^{\pi/2} = \frac{9\pi}{4} - \frac{3\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}
\]

(b) Note that these curves intersect when $\theta = 0$ or $\pi$ and using the work done above, we have

\[
A = 2 \left[ \frac{1}{2} \int_{\pi}^{3\pi/2} (1 - \sin(\theta))^2 \, d\theta - \frac{1}{2} \int_{\pi}^{3\pi/2} (1)^2 \, d\theta \right] = \left[ \frac{3}{2} \theta + 2 \cos(\theta) - \frac{\sin(2\theta)}{4} - \theta \right]_{\pi}^{3\pi/2}
\]

\[
= \left[ \frac{1}{2} \theta + 2 \cos(\theta) - \frac{\sin(2\theta)}{4} \right]_{\pi}^{3\pi/2} = \frac{3\pi}{4} - \left( \frac{\pi}{2} - 2 \right) = \frac{\pi}{4} + 2
\]

(c) Using symmetry we have

\[
A = 2 \left[ \frac{1}{2} \int_{0}^{\pi/4} \left( \sqrt{\sin(2\theta)} \right)^2 \, d\theta \right] = \int_{0}^{\pi/4} \sin(2\theta) \, d\theta = -\frac{\cos(2\theta)}{2} \bigg|_{0}^{\pi/4} = \frac{1}{2}
\]