1. Determine if the series is absolutely convergent, conditionally convergent or divergent, justify your answer:

(a) (10 pts) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}} \]

(b) (10 pts) \[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}e^n}{(n+1)!} \]

(c) (10 pts) \[ \sum_{n=6}^{\infty} \frac{1}{n(\ln n)^2} \]

**Solution:**

(a) (10 pts) Note that \( \sum |a_n| = \sum \frac{1}{n^{2/3}} \) which is a divergent \( p \)-series and note that \( \sum a_n = \sum \frac{(-1)^n}{n^{2/3}} \) is an alternating series with \( b_n = 1/n^{2/3} \) and note that \( b_n \) is decreasing since \( (n+1)^{2/3} > n^{2/3} \) implies \( b_{n+1} < b_n \) and clearly \( \lim_{n \to \infty} b_n = 0 \) and so the original series converges by Alternating Series Test and thus \( \sum \frac{(-1)^n}{n^{2/3}} \) is conditionally convergent.

(b) (10 pts) We use the Ratio Test, note that

\[ \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}e^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{(-1)^n e^n} \right| = \lim_{n \to \infty} \frac{e}{n+2} = 0 < 1 \]

and so \( \sum_{n=0}^{\infty} \frac{(-1)^{n+1}e^n}{(n+1)!} \) is absolutely convergent by Ratio Test.

(c) (10 pts) First note that the series is positive, so it is either absolutely convergent or divergent. We attempt to use the Integral Test. Note that \( f(x) = 1/x \ln^2(x) \) is positive and continuous for \( x \geq 6 \), and decreasing (since the numerator is a constant and \( x \ln^2(x) \to +\infty \) as \( x \to +\infty \)) and note that

\[ \int_6^{\infty} \frac{1}{x(\ln x)^2} \, dx = \int_{\ln(6)}^{\infty} \frac{1}{u^2} \, du \]

and \( \int_{\ln(6)}^{\infty} \frac{1}{u^2} \, du \) is convergent \( p \)-integral (see section 6.6) and so \( \sum_{n=6}^{\infty} \frac{1}{n(\ln n)^2} \) is absolutely convergent by the Integral Test.

2. (a) (10 pts) Find the radius of convergence of \( \sum_{n=1}^{\infty} \frac{3^n(x-3)^n}{\sqrt{n+2}} \). Show all work, justify your answer.

(b) (10 pts) Use the series \( \tan(x) = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \cdots \) to evaluate \( \lim_{x \to 0} \frac{\tan(x) - x - x^3/2}{x^3} \).

(c) (10 pts) Find the \( 2^{nd} \) order Taylor polynomial, \( T_2(x) \), of \( f(x) = \sqrt{2+x} \) centered at \( a = 0 \). Simplify your answer as much as possible.

**Solution:**

(a) (10 pts) First we apply the Ratio Test:

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{3^{n+1}(x-3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+3}}{3^n(x-3)^n} \right| = 3|x-3| \lim_{n \to \infty} \frac{\sqrt{n+2}}{\sqrt{n+3}} = 3|x-3| \lim_{n \to \infty} \sqrt{\frac{1+n}{1+3/n}} = 3|x-3| \]

and now note that \( 3|x-3| < 1 \implies |x-3| < 1/3 \) and so the radius of convergence is \( R = 1/3 \).
(b)(10 pts) Using the series \( \tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \) in the limit, we have

\[
\lim_{x \to 0} \frac{\tan(x) - x - x^3/2}{x^3} = \lim_{x \to 0} \left( \frac{x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots - x - x^3/2}{x^3} \right) = \lim_{x \to 0} \left( \frac{\frac{x^3}{3} + \frac{2x^5}{15} + \cdots}{x^3} \right) = \lim_{x \to 0} \left( \frac{\frac{1}{6} + \frac{2x^2}{15} + \cdots}{x^3} \right) = -\frac{1}{6}
\]

(c)(10 pts) We use the first 3 terms of the Binomial Series of \( f(x) = (2 + x)^{1/2} \), note that

\[
f(x) = (2 + x)^{1/2} = \sqrt{2}(1 + x/2)^{1/2} = \sqrt{2} \sum_{n=0}^{\infty} \binom{1/2}{n} \left( \frac{x}{2} \right)^n = \sqrt{2} \left[ 1 + \frac{1}{2} \cdot \frac{x}{2} + \left( \frac{1}{2} \right) \frac{x^2}{4} + \cdots \right]
\]

and so

\[
T_2(x) = \sqrt{2} \left( 1 + \frac{x}{4} + \frac{(1/2)(1/2 - 1)}{2!} \cdot \frac{x^2}{4} \right) = \sqrt{2} + \frac{\sqrt{2}}{4} x - \frac{\sqrt{2}}{32} x^2
\]

3. (a)(12 pts) Use the Maclaurin series of \( f(x) = \cos(x^2) \) to find a series representation of \( \int_0^1 \cos(x^2) \, dx \).

(b)(8 pts) What is the minimum value of \( n \) that should be used from the series found in part (a) to approximate \( \int_0^1 \cos(x^2) \, dx \) with an error less than 0.005?

Solution: (a)(12 pts) Using the Maclaurin Series of \( \cos(x) \) from the formula sheet, we have

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \implies \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}
\]

thus

\[
\int_0^1 \cos(x^2) \, dx = \int_0^1 \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} \right] \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{4n+1}}{4n+1} \bigg|_{x=0}^{x=1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!}
\]

(b)(8 pts) The above series is an alternating series with \( b_n = 1/(4n + 1)(2n)! \) which is decreasing and the limit goes to 0, so by the Alternating Series Estimation Theorem, we know that the error is no greater than \( b_{N+1} \). Now note that

\[
\sum_{n=0}^{2} \frac{(-1)^n}{(4n+1)(2n)!} = 1 - \frac{1}{10} + \frac{1}{216} + \cdots
\]

and clearly \( b_2 = 1/216 < 1/200 = 0.005 \), thus, we need to use at least two terms of the series or we need to use \( n = 1 \) or greater to achieve an error less than 0.005.

4. (a)(10 pts) Find the 2nd order Taylor polynomial, \( T_2(x) \), of the function \( g(x) = x^{-2} \) centered at \( a = 1 \).

(b)(10 pts) Find an error bound for the approximation \( x^{-2} \approx T_2(x) \) from part (a) if \( 1 \leq x \leq 1.5 \).

Solution: (a)(10 pts) Note that here \( T_2(x) = g(1) + g'(1)(x - 1) + \frac{g''(1)}{2!} (x - 1)^2 \) where

\[
g(1) = 1 \quad \text{and} \quad g'(x) = -2x^{-3} \implies g'(1) = -2 \quad \text{and} \quad g''(x) = 6x^{-4} \implies g''(1) = 6
\]

thus, \( T_2(x) = 1 - 2(x - 1) + \frac{6}{2!}(x - 1)^2 = 1 - 2(x - 1) + 3(x - 1)^2 \).
(b)(10 pts) The error term here is \( R_2(x) = \frac{g^{(3)}(z)}{3!} (x - 1)^3 \) where \( z \) is some number between \( a = 1 \) and \( x \), also note that \( g'''(x) = -24x^{-5} \), so

\[
|R_2(x)| = \left| \frac{g^{(3)}(z)}{3!} (x - 1)^3 \right| = \left| \frac{1}{3!} \cdot \frac{-24}{z^5} \cdot (x - 1)^3 \right| = \frac{4}{|z|^5} |x - 1|^3
\]

and since \( z \) is a number between \( a = 1 \) and \( 1 \leq x \leq 3/2 \), we have that

\[
1 < z < \frac{3}{2} \Rightarrow 0 < \frac{2}{3} < \frac{1}{z} < 1 \Rightarrow 0 < \frac{1}{z^5} < 1 \Rightarrow \frac{4}{|z|^5} < 4
\]

and note that \( 1 \leq x \leq 3/2 \Rightarrow 0 \leq x - 1 \leq 1/2 \Rightarrow |x - 1|^3 \leq 1/8 \) and so we have the following error bound

\[
|R_2(x)| = \frac{4}{|z|^5} |x - 1|^3 < 4 \cdot \frac{1}{8} = \frac{1}{2}
\]