1. The following problems are not related, justify your answers:

(a) (10 pts) Find the exact length of the curve \( y = \ln(\cos(x)) \) for \( 0 \leq x \leq \pi/4 \).

(b) (10 pts) Find the volume of the solid obtained by rotating the region bounded by the curves \( x = \pi/4, x = \pi/2, y = 0 \) and \( y = \cos(x) \) about the \( y \)-axis using the method of cylindrical shells.

(c) (5 pts) Set up (but do not solve) a definite integral to find the volume of the solid obtained by rotating the region bounded by the curves \( x = \pi/4, x = \pi/2, y = 0 \) and \( y = \cos(x) \) about the \( y \)-axis using the method of disks/washers.

**Solution:** (a) (10 pts) Note that \( y' = -\sin(x)/\cos(x) = -\tan(x) \) and recall that \( 1 + \tan^2(x) = \sec^2(x) \), thus,

\[
L = \int_a^b \sqrt{1 + (dy/dx)^2} \, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2(x)} \, dx
\]

\[
= \int_0^{\pi/4} |\sec(x)| \, dx = \ln(\sec(x) + \tan(x))\bigg|_0^{\pi/4} = \ln(\sqrt{2} + 1) - \ln(1) = \ln(\sqrt{2} + 1)
\]

(b) (10 pts) Here note that \( \Delta V = 2\pi rh\Delta x \) where \( r = x \) and \( h = \cos(x) \), thus \( V = \int_{\pi/4}^{\pi/2} 2\pi x \cos(x) \, dx \) and using integration by parts with \( u = x \Rightarrow du = dx \) and \( dv = \cos(x) \Rightarrow v = \sin(x) \) and so

\[
V = \int_{\pi/4}^{\pi/2} 2\pi x \cos(x) \, dx \quad \text{IBP} \quad 2\pi \left[ x \sin(x)\bigg|_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \sin(x) \, dx \right]
\]

\[
= 2\pi \left( \frac{\pi}{2} - \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} \right) + \cos(x)\bigg|_{\pi/4}^{\pi/2} = \frac{\pi^2}{4} - \frac{\pi\sqrt{2}}{4} - \pi\sqrt{2}
\]

(c) (5 pts) Here \( \Delta V = \pi[R^2 - r^2] \Delta y \) where \( R = \arccos(y) \) and \( r = \pi/4 \) so we have \( V = \int_0^{\sqrt{2}/2} \pi [\arccos^2(y) - (\pi/4)^2] \, dy \)

2. The following problems are not related, justify your answers:

(a) (10 pts) Find the area of the surface obtained by rotating \( 3x = y^3 \) for \( 0 \leq y \leq 2 \) about the \( y \)-axis.

(b) (15 pts) Suppose that a thin metal plate is defined by the region bounded by \( f(x) = e^x \) and \( g(x) = 1 \) for \( 0 \leq x \leq 1 \). Further, suppose that the mass of the plate is \( M = 4e - 8 \). Find the moment about the \( x \)-axis, \( M_x \).

**Solution:** (a) (10 pts) Note that \( x = y^3/3 = g(y) \) and \( g'(y) = y^2 \), and so, in terms of \( y \), we have

\[
SA = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2} \, dy = \frac{2\pi}{3} \int_0^2 y^3\sqrt{1 + y^4} \, dy
\]

\[
\underset{u = 1 + y^4}{=} \frac{2\pi}{3} \frac{1}{4} \int_1^{17} \sqrt{u} \, du = \frac{\pi}{6} \frac{2}{3} y^{3/2}\bigg|_1^{17} = \frac{\pi}{9} (17^{3/2} - 1)
\]

(in terms of \( x \), we have \( f(x) = (3x)^{1/3} \) and \( f'(x) = (3x)^{-2/3} \) and so

\[
SA = \int_a^b 2\pi x \sqrt{1 + f'(x)^2} \, dx = \int_0^{8/3} 2\pi x \sqrt{1 + (3x)^{-4/3}} \, dx = \frac{2\pi}{3^{2/3}} \int_0^{8/3} x^{1/3} \sqrt{(3x)^4 + 1} \, dx
\]

and now if we let \( u = (3x)^{4/3} + 1 \) then we see, as before, that \( SA = \pi(17^{7/2} - 1)/9. \))
4. Determine if the series converges or diverges, find the value of the series if it converges, show all work:

\[ M = \rho \int_0^1 (f(x) - g(x)) \, dx = \rho \int_0^1 (e^x - 1) \, dx = \rho \left[ (e^x - x) \right]_0^1 = \rho (e^1 - 1 - (e^0 - 0)) = \rho (e - 2) \]

and note that we are given that \( M = 4e - 8 \) so \( \rho (e - 2) = 4e - 8 = 4(e - 2) \) implies \( \rho = 4 \). Now to find \( M_x \), note that

\[ M_x = \frac{\rho}{2} \int_a^b (f^2(x) - g^2(x)) \, dx = \frac{4}{2} \int_0^1 (e^{2x} - 1) \, dx = 2 \left[ \left( \frac{e^{2x}}{2} - x \right) \right]_0^1 = 2 \left[ \left( \frac{e^2}{2} - 1 \right) - \left( \frac{1}{2} - 0 \right) \right] = 2 \left[ \frac{e^2}{2} - \frac{3}{2} \right] = e^2 - 3 \]

3. The following problems are not related, justify your answers:

(a) (10 pts) Does the sequence \( \{ \cos \left( \frac{n}{2} \right) \} \) converge? Why or why not? If it converges, find the limit.

Solution: (a) (10 pts) Note that \( 2n - 1 \) is always odd for \( n = 1, 2, 3, \ldots \) and so

\[ \left\{ \cos \left( \frac{2n-1}{2} \right) \right\} \xrightarrow{n \to \infty} \cos(\pi/2), \cos(3\pi/2), \cos(5\pi/2), \ldots = 0, 0, 0, \ldots \rightarrow 0 \]

and so the sequence converges to 0.

(b) (15 pts) Let \( \sigma_N \) be the \( N \)th partial sum of the series \( \sum_{n=1}^\infty (s_n - s_{n+1}) \), then note that

\[ \sigma_N = (s_1 - s_2) + (s_2 - s_3) + (s_3 - s_4) + \cdots + (s_{N-1} - s_N) + (s_N - s_{N+1}) = s_1 - s_{N+1} \]

and now, noting that \( s_1 = a_1 = 1 \) and that \( \lim_{N \to \infty} s_N = \sum_{i=1}^\infty a_i = 1/4 \), we see that

\[ \sum_{n=1}^\infty (s_n - s_{n+1}) = \lim_{N \to \infty} \sigma_N = \lim_{N \to \infty} (s_1 - s_{N+1}) = s_1 - \lim_{N \to \infty} s_{N+1} = a_1 - \sum_{i=1}^\infty a_i = 1 - \frac{1}{4} = \frac{3}{4} \]

4. Determine if the series converges or diverges, find the value of the series if it converges, show all work:

(a) (8 pts) \[ \sum_{n=0}^\infty \frac{2^{1-n}}{3^n} \]

(b) (8 pts) \[ \sum_{n=1}^\infty \frac{\sqrt{3}}{8} \]

(c) (9 pts) Let \( \{b_n\}_{n=1}^\infty \) be a sequence of numbers between 0 and 1 (0 < \( b_n \) < 1 for \( n > 0 \)). Now consider the sequence \( a_1 = 1 \) and \( a_n = b_n a_{n-1} \) for \( n \geq 2 \), does this sequence \( \{a_n\}_{n=1}^\infty \) converge or diverge? If the sequence does converge, what is its limit? Justify your answers.

Solution: (a) (8 pts) Here note that we have a geometric series since

\[ \sum_{n=0}^\infty \frac{2^{1-n}}{3^n} = \sum_{n=0}^\infty \frac{2^{1-n}}{3^n} = \sum_{n=0}^\infty 2 \left( \frac{1}{6} \right)^n = 2 + 2 \left( \frac{1}{6} \right)^1 + 2 \left( \frac{1}{6} \right)^2 + 2 \left( \frac{1}{6} \right)^3 + \cdots = \frac{2}{1 - 1/6} = 12/5. \]

and so we have a geometric series that converges to 12/5.
(b) (8 pts) Here using the Test for Divergence, we see that

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{4^{1/n}}{8} = \frac{1}{8} \neq 0
\]

and so the series diverges by the Test for Divergence.

(c) (9 pts) Note that since \( a_n = b_n a_{n-1} \) for \( n \geq 2 \) we see that \( a_n = b_n b_{n-1} \cdots b_2 a_1 \) for \( n \geq 2 \) and, now, since \( a_1 = 1 > 0 \) and \( b_n > 0 \) for all \( n \) we see that \( a_n = b_n b_{n-1} \cdots b_2 a_1 > 0 \) for \( n \geq 2 \) and thus

\[
0 < b_n < 1 \implies 0 < b_n a_{n-1} < a_{n-1} \implies 0 < a_n < a_{n-1} \text{ for } n \geq 2
\]

thus we see that the sequence \( \{a_n\} \) is **decreasing** and **bounded below** by 0 and so, by the Monotonic Sequence Theorem, \( \{a_n\} \) converges. Now note that using some rigorous math we can show that 0 is the *greatest lower bound* of the sequence \( \{a_n\} \) and so the sequence converged to 0.