1. The following are not related:

(a) (7 pts) Differentiate \( y = \int_{2x}^{x^2} \frac{3 + t}{t^2 + 5} \, dt \) [Do not simplify.]

(b) (6 pts) Evaluate the integral \( \int (1 + \cot^2 x) \, dx \)

(c) (6 pts) Evaluate the integral \( \int \frac{7x^3 + 5x^2 - 3}{\sqrt{x}} \, dx \)

(d) (6 pts) Evaluate the integral \( \int_1^3 |x^2 - 1| \, dx \)

Solution:

(a) 
\[
y = \int_1^2 1 + 5t^2 \, dt + \int_1^{2/x} 3 + t \, dt + \int_1^{2/x} 3 + t \, dt = -\int_1^{2/x} 1 + 5t^2 \, dt + \int_1^{2/x} 3 + t \, dt
\]
\[
\frac{dy}{dx} = -\frac{3 + 2x}{1 + 5(2x)^2} (2) + \frac{3 + 2/x}{1 + 5(2x)^2} \left( -\frac{2}{x^2} \right)
\]

(b) \( \int (1 + \cot^2 x) \, dx = \int \csc^2 x \, dx = -\cot x + C \)

(c) 
\[
\int \frac{7x^3}{x^{1/4}} + \frac{5x^2}{x^{1/4}} - \frac{3}{x^{1/4}} \, dx = \int 7x^{11/4} + 5x^{3/4} - 3x^{-1/4} \, dx = \int 7x^{11/4} + 5x^{7/4} - 3x^{-1/4} \, dx
\]
\[
= 7 \left( \frac{4}{15} \right) x^{15/4} + 5 \left( \frac{4}{11} \right) x^{11/4} - 3 \left( \frac{4}{3} \right) x^{3/4} + C
\]
\[
= \frac{28}{15} x^{15/4} + \frac{20}{11} x^{11/4} - 4x^{3/4} + C
\]

(d) Since \( x^2 - 1 \) is only negative on the interval \((-1, 1)\) and this integral is being evaluated from \((1, 3)\), we may drop the absolute value symbols:
\[
\int_1^3 x^2 - 1 \, dx = \frac{1}{3} x^3 - x \bigg|_1^3 = \frac{20}{3}
\]

2. The following problems are not related:

(a) (12 pts) The function \( f(x) = \frac{1}{2} x^3 - 2x^2 + 3x + 5 \) has two local extrema. Estimate the value of \( x \) where one of the local extreme values of \( f(x) \) occur using one iteration of Newton’s method (in other words, find \( x_2 \)). Use \( x_1 = 0 \) as an initial approximation.

(b) (12 pts) 

i. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm respectively, if two sides of the rectangle lie along the legs.

ii. How do you know your answer is a maximum? Justify your answer based on the theories of this class.
3. (a) (6 pts) Using the definition for area using right hand endpoints, 

\[ A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x] \]

find an expression for the area under the curve \( y = -3x^2 + 6x \) from 0 to 2 as a limit.

Note: The following formulas may be useful:

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2
\]

(b) (6 pts) Evaluate the limit.

(c) (6 pts) Now express the area as an integral and find the average value, \( f_{avg} \).

(d) (6 pts) Find all \( c \) between \( x = 0 \) and \( x = 2 \) so that \( f(c) = f_{avg} \).

Solution:

(a) \( \Delta x = \frac{2}{n}, x_i = \frac{2i}{n} \). So we can express the area as:

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ -3 \left( \frac{2i}{n} \right)^2 + 6 \left( \frac{2i}{n} \right) \right] \left( \frac{2}{n} \right)
\]

(b) Evaluating the limit in part (a) gives us:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left[ -\frac{24i^2}{n^3} + \frac{24i}{n^2} \right] = \lim_{n \to \infty} -\frac{24}{n^3} \sum_{i=1}^{n} i^2 + \frac{24}{n^2} \sum_{i=1}^{n} i
\]

\[
= \lim_{n \to \infty} \left[ -\frac{24}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right) + \frac{24 n(n+1)}{2n^2} \right]
\]

\[
= \lim_{n \to \infty} \left[ -\frac{4}{n^3} (2n^3 + 3n^2 + n) + 12 + \frac{12}{n} \right]
\]

\[
= \lim_{n \to \infty} \left[ -8 - \frac{12}{n} - \frac{4}{n^2} + 12 + \frac{12}{n} \right] = -8 + 12 = 4
\]

(c) \( f(x) = f_0^2 - 3x^2 + 6x \ d x \). So the average value of the integral is given by:

\[
f_{ave} = \frac{1}{2} \left( -x^3 + 3x^2 \big|_0^2 \right) = \frac{1}{2} (-8 + 12) = 2
\]

(d) \( f(c) = -3c^2 + 6c \). So we need \(-3c^2 + 6c = 2 \implies -3c^2 + 6c - 2 = 0 \implies 3c^2 - 6c + 2 = 0 \). Using the quadratic formula we have \( c = 1 \pm \sqrt{\frac{5}{3}} \).
4. Let the function \( f \) be defined by \( f(x) = \int_{\frac{1}{4}}^{x} \frac{1}{t} \, dt \) for \( x > 0 \).

(a) (5 pts) What is \( f(1) \)? What is \( f'(x) \)? What is \( f'(1) \)?
(b) (5 pts) \( f \) is differentiable. Why?
(c) (6 pts) Show that \( f'(5x) = f'(x) \).
(d) (5 pts) Using the definition of \( f \), show that \( f(x + h) - f(x) = \int_{x}^{x+h} \frac{1}{t} \, dt \)
(e) (6 pts) Now suppose \( h(x) = \int_{0}^{\cos(x-2)} 3t^2 \, dt \) and \( f(s) = \int_{\pi}^{4s} h(x) \, dx \).
Find \( f''(1/2) \).

Solution:
(a) $f(1) = \int_{1}^{1} \frac{1}{t} \, dt = 0$

\[ f'(x) = \frac{1}{x} \]

\[ f'(1) = \frac{1}{1} = 1 \]

(b) \( f \) is differentiable because \( \frac{1}{t} \) is continuous on its domain.

(c) $f(5x) = \int_{1}^{5x} \frac{1}{t} \, dt$

\[ \Rightarrow f'(5x) = \frac{1}{5x}(5) = \frac{1}{x} = f'(x) \]

(d) \( f(x + h) = \int_{1}^{x+h} \frac{1}{t} \, dt \), \( f(x) = \int_{1}^{x} \frac{1}{t} \, dt \). So therefore,

\[ f(x + h) - f(x) = \int_{1}^{x+h} \frac{1}{t} \, dt - \int_{1}^{x} \frac{1}{t} \, dt = \int_{1}^{x+h} \frac{1}{t} \, dt + \int_{x}^{1} \frac{1}{t} \, dt \]

\[ \Rightarrow f(x + h) - f(x) = \int_{x}^{x+h} \frac{1}{t} \, dt \]

(e) \( f'(s) = 4h(4s) \).

Then \( f''(s) = 4h'(4s) = 4 \cdot 3 \cos^2(4s - 2)(-4 \sin(4s - 2)) = -48 \cos^2(4s - 2) \sin(4s - 2) \)

\[ f''(1/2) = -48 \cos(0) \sin(0) = 0 \]