1. For this problem, suppose \( f(x) = 2 \cos x \) and \( g(x) = \frac{1}{x^2 - 1} \).

(a) (6 pts) Find \((g \circ f)(x)\).

(b) (6 pts) What is the domain of \((g \circ f)(x)\)?

(c) (8 pts) Suppose we let \( h(x) = \begin{cases} f(x), & \text{if } x > 2\pi \\ g(x), & \text{if } x \leq 2\pi \end{cases} \), are there any values of \( x \) for which \( h(x) \) is not continuous? Justify your answer. What type of discontinuities does \( h(x) \) have (i.e. jump, removable, or infinite), if any?

Solution:

(a)(6 pts) \((g \circ f)(x) = g(f(x)) = g(2 \cos(x)) = \frac{1}{4 \cos^2(x) - 1}\)

(b)(6 pts) The domain of \( f(x) \) is all real \( x \) except for values of \( x \) where \( 4 \cos^2(x) - 1 = 0 \).

\[
\cos^2 x = 1/4 \\
\cos x = \pm 1/2 \\
\Rightarrow x = \ldots -\pi, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots
\]

(c)(8 pts) Note that for \( x > 2\pi \), \( h(x) \) is continuous since \( f(x) \) is well defined and continuous for all \( x \). At \( x = 2\pi \), we need check for continuity. Note that

\[
\lim_{x \to 2\pi^+} h(x) = \lim_{x \to 2\pi^-} 2 \cos x = 2 \cos(2\pi) = 2 \quad \text{and} \quad \lim_{x \to 2\pi^-} h(x) = \lim_{x \to 2\pi^-} \frac{1}{x^2 - 1} = \frac{1}{(2\pi)^2 - 1} \neq 2
\]

Therefore there is a jump discontinuity at \( x = 2\pi \). When \( x < 2\pi \), \( h(x) = \frac{1}{x^2 - 1} \) and so \( h(x) \) has infinite discontinuities at \( x = \pm 1 \).

2. Evaluate the following limits and show all supporting work. If a limit does not exist, clearly state that fact and explain your reasoning. (Note: You may not use l’Hopital’s Rule.)

(a) (4 pts) \( \lim_{{x \to 1}} \frac{x^2 + x - 2}{x^2 - 4x + 3} \)

(b) (4 pts) \( \lim_{{x \to \infty}} 2x - \sqrt{4x^2 - 5x} \)

(c) (4 pts) \( \lim_{{x \to 0^+}} \sqrt{x} \cos \frac{\pi}{x} \)

(d) (4 pts) \( \lim_{{x \to 0^+}} \frac{x}{x - |x|} \)

(e) (4 pts) \( \lim_{{x \to \infty}} \sqrt{\frac{4x^2 - x}{x^2 + 9}} \)

Solution:

(a) (4 pts) \( \lim_{{x \to 1}} \frac{x^2 + x - 2}{x^2 - 4x + 3} = \lim_{{x \to 1}} \frac{(x - 1)(x + 2)}{(x - 3)(x - 1)} = \lim_{{x \to 1}} \frac{x + 2}{x - 3} = \frac{3}{2} \)
(b) \( \lim_{x \to -\infty} 2x - \sqrt{4x^2 - 5x} = \lim_{x \to -\infty} 2x - |x| \sqrt{4 - \frac{5}{x}} = \lim_{x \to -\infty} 2x + x \sqrt{4 - \frac{5}{x}} \)

\[ \lim_{x \to -\infty} 2x + 2x = \lim_{x \to -\infty} 4x = -\infty \]

(c) \( \lim_{x \to 0^+} \sqrt{x} \cos \frac{\pi}{x} \)

We may bound cosine: \(-1 \leq \cos \frac{\pi}{x} \leq 1\). Then \(-\sqrt{x} \leq \sqrt{x} \cos \frac{\pi}{x} \leq \sqrt{x}\). Now we can apply the Squeeze Theorem:

\[ \lim_{x \to 0^+} -\sqrt{x} \leq \lim_{x \to 0^+} \sqrt{x} \cos \frac{\pi}{x} \leq \lim_{x \to 0^+} \sqrt{x} \]

\[ 0 \leq \lim_{x \to 0^+} \sqrt{x} \cos \frac{\pi}{x} \leq 0 \]

\[ \implies \lim_{x \to 0^+} \sqrt{x} \cos \frac{\pi}{x} = 0 \]

(d) \( \lim_{x \to 0^-} \frac{x}{x - |x|} = \lim_{x \to 0^-} \frac{x}{x - x} = \frac{1}{2} \)

(e) \( \lim_{x \to \infty} \sqrt{\frac{4x^2 - x}{x^2 + 9}} = \sqrt{\lim_{x \to \infty} \frac{4x^2 - x}{x^2 + 9}} = \sqrt{4} = 2 \)

3. (a) (5 pts) Given the function \( f(x) = 3^{-x} \cos(10x) \). Is \( f \) a continuous function of \( x \)? Justify why or why not.

(b) (5 pts) Does \( f(x) = 3^{-x} \cos(10x) \) have a real root? Justify why or why not.

(c) (5 pts) Use continuity to evaluate: \( \lim_{x \to \pi} \sin(x + \sin x) \).

(d) (5 pts) Use the definition of the derivative to show that \( b(x) = \sqrt{x} + 1 \) is an increasing function.

**Solution:**

(a) (5 pts) Yes, \( \frac{1}{3^x} \) is continuous for all \( x \in (-\infty, \infty) \) and \( \cos(10x) \) is also continuous for all \( x \in (-\infty, \infty) \), therefore the product of those two continuous functions is also continuous on \( x \in (-\infty, \infty) \).

(b) (5 pts) Since this function is continuous everywhere, we apply the Intermediate Value Theorem:

\[ f(\pi/10) = \frac{1}{3^{\pi/10}} \cos(\pi) = -\frac{1}{3^{\pi/10}} < 0 \]

\[ f(\pi) = \frac{1}{3^\pi} \cos(10\pi) = \frac{1}{3^\pi} > 0 \]

Therefore \( f(x) \) must have a root somewhere in the interval \( (\frac{\pi}{10}, \pi) \). [Note that \( f(x) \) actually has infinitely many roots.]

(c) (5 pts) \( \lim_{x \to \pi} \sin(x + \sin x) = \sin(\lim_{x \to \pi} (x + \sin x)) = \sin(\pi + 0) = 0 \)

(d) (5 pts) Using the definition of a derivative:

\[ b'(x) = \lim_{h \to 0} \frac{b(x+h) - b(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} + x + h - 1 - (\sqrt{x} + x - 1)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x} + h}{h} \]

\[ = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x} + h}{h} \cdot \frac{\sqrt{x} + h + (\sqrt{x} - h)}{\sqrt{x} + h + (\sqrt{x} - h)} = \lim_{h \to 0} \frac{x + h - (\sqrt{x} - h)^2}{h(\sqrt{x} + h + \sqrt{x} - h)} \]

\[ = \lim_{h \to 0} \frac{1 + 2\sqrt{x} + h}{\sqrt{x} + h + \sqrt{x} - h} = \frac{1 + 2\sqrt{x}}{2\sqrt{x}} = 1 + \frac{1}{2\sqrt{x}} \]

So \( b'(x) = 1 + \frac{1}{2\sqrt{x}} > 0 \) for all \( x \) in its domain. Therefore this function must be increasing for all \( x \) in its domain.
4. (a) (7 pts) Use the limit definition of the derivative to find the slope of \( f(x) = 3x^2 - 10x - 7 \) at any point \( x \).

(b) (7 pts) Find an equation of the tangent line to the parabola \( f(x) = 3x^2 - 10x - 7 \) whose slope is \( m = -8 \).

(c) (6 pts) If \( s(t) = 3t^2 - 10t - 7 \) for \( t \geq 0 \) describes the position of an object (in feet) at time \( t \), find the average velocity of the object from \( t = 1 \) second to \( t = 2 \) seconds.

Solution:

(a) (7 pts)

\[
\lim_{h \to 0} \frac{3(x + h)^2 - 10(x + h) - 7 - (3x^2 - 10x - 7)}{h} = \lim_{h \to 0} \frac{6xh + 3h^2 - 10h}{h} = \lim_{h \to 0} 6x + 3h - 10 = 6x - 10
\]

So therefore, \( f'(x) = 6x - 10 \) describes the slope at any point \( x \).

(b) (7 pts) \(-8 = 6x - 10 \implies x = \frac{1}{3} \). Then \( f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^2 - 10\left(\frac{1}{3}\right) - 7 = -10 \). To find the equation of the tangent line, we use the point-slope formula:

\[
y - (-10) = -8(x - \frac{1}{3}) \implies y = -8x + \frac{22}{3}
\]

(c) (6 pts)

\[
v_{ave} = \frac{s(2) - s(1)}{2 - 1} = \frac{-15 - (-14)}{1} = -1
\]

Therefore, the average velocity between 1 and 2 seconds is -1 ft/sec.

5. The following parts are not related:

(a) (6 pts) For what values of \( x \) does the graph of \( f(x) = x + 2 \sin x \) have a horizontal tangent?

(b) (6 pts) Find the first and second derivatives of: \( G(r) = \sqrt{r} + \frac{3}{\sqrt{r}} \).

(c) (8 pts) Find the \( n \)th derivative of each function by calculating the first few derivatives and observing the pattern that occurs:

i. \( f(x) = x^n \)

ii. \( f(x) = \frac{1}{x} \)

Solution:

(a) (6 pts) We must find when \( f'(x) = 0 \).

\[
f'(x) = 1 + 2 \cos x = 0 \implies \cos x = -\frac{1}{2} \implies x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, ...
\]

(b) (6 pts) \( G(r) = r^{1/2} + r^{1/3} \)

\[
G'(r) = \frac{1}{2} r^{-1/2} + \frac{1}{3} r^{-2/3}
\]

\[
G''(r) = -\frac{1}{4} r^{-3/2} - \frac{2}{9} r^{-5/3}
\]

(c) (8 pts)

(i) \( f^{(n)}(x) = n(n-1)(n-2) \cdots (n-(n-1))x^{n-n} = n! \)

(ii) \( f^{(n)}(x) = (-1)^n n! x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}} \)