1. (20 pts) Consider the function \( f(x) = \sqrt{2 - x} \).

(a) Find the linear approximation for \( f(x) \) at \( x = -2 \).

(b) Use your approximation from part (a) to estimate the value of \( \sqrt{4.1} \).

(c) Compute \( f''(x) \). Use \( f'' \) to explain whether your approximation from (b) is an overestimate or an underestimate.

Solution:

(a) Notice that \( f'(x) = -\frac{1}{2}(2 - x)^{-1/2} = -\frac{1}{2\sqrt{2 - x}} \), \( f'(-2) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4} \), and \( f(-2) = \sqrt{4} = 2 \).

The linearization of \( f(x) \) at \( x = -2 \) is given by

\[
L(x) = f(-2) + f'(-2)(x - (-2))
= 2 - \frac{1}{4}(x + 2).
\]

(b) \( \sqrt{4.1} = \sqrt{2 - (-2.1)} = f(-2.1) \approx L(-2.1) = 2 - \frac{1}{4}(-2.1 + 2) = 2 - \frac{1}{4}(-0.1) = 2 + \frac{1}{4} \cdot \frac{1}{10} = \frac{81}{40} \).

(c) \( f''(x) = -\frac{1}{4}(2 - x)^{-3/2} = -\frac{1}{4(2 - x)^{3/2}} \). Since, \( f''(x) < 0 \) for all \( x \) in the domain, the function is concave down on its whole domain. That is, the graph of \( y = f(x) \) lies below its tangent lines, so the approximation in (b) must be an overestimate.

2. (24 pts) A rocket is launched vertically and is tracked by a radar station located on the ground 4 miles from the launch pad. If the angle between the ground and the line of sight from the radar station to the rocket is increasing at a rate of 0.05 radians per second, what is the speed of the rocket when the angle is \( \frac{\pi}{3} \) radians? Assume the ground is horizontal and flat. A complete answer should include a labeled diagram and the correct units.

Solution:

We know that \( \frac{d\theta}{dt} = 0.05 \) radians per second. We want to find \( \frac{dh}{dt} \) when \( \theta = \frac{\pi}{3} \).

Notice that \( \theta \) and \( h \) are related by

\[
\tan \theta = \frac{h}{4}.
\]
Implicitly differentiating, we find that
\[ \sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{4} \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{4}{\cos^2 \theta} \frac{d\theta}{dt}. \]

When \( \theta = \frac{\pi}{3} \), \( \cos \theta = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2} \). Thus we have
\[ \frac{dh}{dt} = \frac{4}{\left( \frac{1}{2} \right)^2} \cdot 0.05 = 16 \cdot 0.05 = 0.8. \]

That is, the rocket is moving at a speed of 0.8 miles per second, or 0.8 \( \text{mi sec}^{-1} \times 3600 \text{ sec hr}^{-1} = 2880 \text{ miles per hour.} \)

3. (35 pts) Consider the function \( f(x) = \frac{\sin x}{\cos x + 2} \), whose second derivative is \( f''(x) = \frac{2 \sin x (\cos x - 1)}{(\cos x + 2)^3} \).

You must justify your answers for each part of this problem.

(a) Show that \( f'(x) = \frac{2 \cos x + 1}{(\cos x + 2)^2} \).

(b) What is the domain of \( f(x) \)? Is \( f(x) \) even, odd, or neither?

(c) Find the \( x \) and \( y \) intercepts of \( f(x) \).

(d) Find the vertical and horizontal asymptotes of \( f(x) \), if they exist.

(e) Find the intervals of increase and decrease of \( f(x) \) for \( 0 \leq x \leq 2\pi \). What are the coordinates of any local extrema of \( f(x) \) in this interval?

(f) Where is \( f(x) \) concave up for \( 0 \leq x \leq 2\pi \)? Where is \( f(x) \) concave down?

(g) Sketch \( f(x) \) for \( 0 \leq x \leq 2\pi \).

Solution:

(a) To find \( f'(x) \), we use the quotient rule:
\[ f'(x) = \frac{(\cos x + 2)(\cos x) - (\sin x)(-\sin x)}{(\cos x + 2)^2} = \frac{\cos^2 x + 2 \cos x + \sin^2 x}{(\cos x + 2)^2} = \frac{2 \cos x + 1}{(\cos x + 2)^2}, \]
where we used \( \sin^2 x + \cos^2 x = 1 \).

(b) The domain of \( f(x) \) is all real numbers since the denominator can never be zero.

\( f(x) \) is an odd function: \( f(-x) = \frac{\sin (-x)}{\cos (-x) + 2} = \frac{-\sin x}{\cos x + 2} = -f(x) \), where we used the fact that \( \sin x \) is odd (so \( \sin (-x) = -\sin x \)) and \( \cos x \) is even (so \( \cos (-x) = \cos x \)).

(c) The \( x \) intercepts are found by setting \( y = 0 \) (where \( y = f(x) \)):
\[ \frac{\sin x}{\cos x + 2} = 0 \implies \sin x = 0 \implies x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots \]
In other words, the $x$ intercepts are $x = n\pi$, where $n$ is an integer.

The $y$ intercepts are found by setting $x = 0$:

$$y = \frac{\sin 0}{\cos 0 + 2} = \frac{0}{3} \implies y = 0.$$

That is, the $y$ intercept is $y = 0$.

(d) There are no vertical or horizontal asymptotes.

(e) The critical points of $f(x)$ occur where $f'(x) = 0$ or $f'(x)$ DNE. Since $f'(x) = \frac{2\cos x + 1}{(\cos x + 2)^2}$, we have that $f'(x) = 0$ when $2\cos x + 1 = 0$:

$$2\cos x + 1 = 0 \implies \cos x = -\frac{1}{2} \implies x = \frac{2\pi}{3} \text{ and } x = \frac{4\pi}{3}.$$

$f'(x)$ always exists, so $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ are the only critical points in $[0, 2\pi]$.

We check the sign of $f'$ in the intervals $[0, \frac{2\pi}{3}), (\frac{2\pi}{3}, \frac{4\pi}{3})$, and $(\frac{4\pi}{3}, 2\pi]$:

$$\left[0, \frac{2\pi}{3}\right): \quad f' > 0$$
$$\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right): \quad f' < 0$$
$$\left(\frac{4\pi}{3}, 2\pi\right]: \quad f' > 0$$

It follows that $f$ is increasing on $[0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi]$ and decreasing on $(\frac{2\pi}{3}, \frac{4\pi}{3})$. There is a local maximum at $x = \frac{2\pi}{3}$, with value

$$f\left(\frac{2\pi}{3}\right) = \frac{\sin \left(\frac{2\pi}{3}\right)}{\cos \left(\frac{2\pi}{3}\right) + 2} = \frac{\sqrt{3}}{2} + \frac{1}{2} + 2 = \frac{\sqrt{3}}{3}.$$ 

There is a local minimum at $x = \frac{4\pi}{3}$, with value

$$f\left(\frac{4\pi}{3}\right) = \frac{\sin \left(\frac{4\pi}{3}\right)}{\cos \left(\frac{4\pi}{3}\right) + 2} = \frac{-\sqrt{3}}{2} + \frac{1}{2} + 2 = -\frac{\sqrt{3}}{3}.$$ 

(f) Since $f''(x) = \frac{2\sin x (\cos x - 1)}{(\cos x + 2)^3}$, we see that $f''(x)$ always exists and is zero when

$$2\sin x (\cos x - 1) = 0 \quad \implies \quad \sin x = 0 \text{ or } \cos x - 1 = 0 \quad \implies \quad x = 0, \pi, 2\pi.$$

We check the sign of $f''$ in the intervals $(0, \pi)$ and $(\pi, 2\pi)$:

$(0, \pi): \quad f'' < 0$
$(\pi, 2\pi): \quad f'' > 0$

It follows that $f$ is concave down on $(0, \pi)$ and concave up on $(\pi, 2\pi)$. 
4. (15 points) Let \( y = x^5 + 2x^3 + 5x + 2 \) on the interval \((-1, 1)\). Show that at least one tangent line to the curve is parallel to the line \( y = 8x + 3 \). If you use any theorems, you must state them and show that their conditions are satisfied.

**Solution:**
We want to show that there is a tangent line to \( y = f(x) \) at some point in \((-1, 1)\) with slope \( m = 8 \) (since the line \( y = 8x + 3 \) has slope 8). We use the Mean Value Theorem, which states that if \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a number \( c \) in \((a, b)\) such that

\[
 f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

In our case, \( f(x) \) is a polynomial, so it is continuous and differentiable for all real numbers. In particular, \( f \) is continuous on \([-1, 1]\) and differentiable on \((-1, 1)\). From MVT, there is a \( c \) in \((-1, 1)\) such that

\[
 f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{10 - (-6)}{2} = 8.
\]

That is, there is a \( c \) in \((-1, 1)\) for which \( f'(c) = 8 \), meaning that there is a point in \((-1, 1)\) at which the tangent line has slope \( m = 8 \).

5. (6 points) For this question, answer with the word True or False. Do not write T or F. You do not need to show any work for this problem.

(a) If \( f'(c) = 0 \), then there is a local maximum or a local minimum at \( x = c \).

(b) If \( f(x) \) and \( g(x) \) are increasing on an interval \( I \), then the product \( f(x)g(x) \) is also increasing on \( I \).

(c) If \( f'(x) = g'(x) \) for \(-1 < x < 1\), then \( f(x) = g(x) \) for \(-1 < x < 1\).

**Solution:**

(a) False. Consider \( f(x) = x^3 \). Then \( f'(x) = 3x^2 \), and \( f'(x) = 0 \) when \( x = 0 \). However, \( f(x) \) has neither a local maximum nor a local minimum at \( x = 0 \).

(b) False. Suppose \( f(x) = x + 1 \) and \( g(x) = x - 1 \) on \([-1, 0]\). Then \( f'(x) = g'(x) = 1 \), so both functions are increasing on \([-1, 0]\). However, \( f(x)g(x) = x^2 - 1 \), so \( (fg)' = 2x \). It follows that \( f(x)g(x) \) must be decreasing when \( x < 0 \), and in particular on \([-1, 0]\).

(c) False. If \( f'(x) = g'(x) \), then \( f(x) \) and \( g(x) \) can differ by a constant. (See Corollary 7 in Section 3.2.)