1. (2 pts each) True/False
   
   (a) (T/F) If \( f \) is undefined at \( x = c \), then the limit of \( f(x) \) as \( x \) approaches \( c \) does not exist.  
   (b) (T/F) If the limit of \( f(x) \) as \( x \) approaches \( c \) is 0, then there must be a number \( k \) such that \( f(k) < 0.0001 \).  
   (c) (T/F) \( \lim_{x \to 0} \sin \left( \frac{|x|}{x} \right) = 0 \)  
   (d) (T/F) If \( f \) is an even function and \( \lim_{x \to 2^-} f(x) = 7 \) then \( \lim_{x \to -2^-} f(x) = 7 \)

   **Solution:**  
   (a) FALSE  
   (b) TRUE  
   (c) FALSE  
   (d) FALSE

2. Evaluate the following limits, you may not use l’Hospital’s Rule, justify your answers:

   (a) (7 pts) \( \lim_{x \to 1} \frac{\sin(2x)}{\sin(3x)} \)
   (b) (7 pts) \( \lim_{x \to 1^+} \frac{\sqrt{2x(x-1)}}{|x-1|} \)
   (c) (7 pts) \( \lim_{x \to 0} \sqrt{x} \sin \left( \frac{1}{x} \right) \)
   (d) \( f(b^2 + 1) = \) given that \( f(x) = \begin{cases} |x| + 1, & \text{if } x < 1 \\ -x + 1, & \text{if } x \geq 1 \end{cases} \)

   **Solution:**  
   (a) \( \lim_{x \to 1} \frac{\sin(2x)}{\sin(3x)} = \frac{\sin(2)}{\sin(3)} \)
   (b) \( \lim_{x \to 1^+} \frac{\sqrt{2x(x-1)}}{|x-1|} \) is an indeterminate form \( \frac{0}{0} \) but we can cancel because of the one-sided limit, that is, as \( x \to 1^+ \), \( |x - 1| = (x - 1) \) and so \( \lim_{x \to 1^+} \frac{\sqrt{2x(x-1)}}{|x-1|} = \lim_{x \to 1^+} \frac{\sqrt{2x(x-1)}}{(x-1)} = \lim_{x \to 1^+} \sqrt{2x} = \sqrt{2} \).  
   (c) The limit does not exist because the left hand limit does not exist.
   (d) The quantity \( b^2 + 1 \) is always greater or equal to 1 so we simply have \( f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2 \).

3. (18 pts) For what value(s) of \( k \) is the function \( f(x) = \begin{cases} \sin(kx), & \text{if } x \leq 0 \\ 3x, & \text{if } x > 0 \end{cases} \) continuous at \( x = 0 \). A complete answer will include the definition of continuity.

   **Solution:** We have \( f(0) = 0 \) and so to be continuous both one-sided limits must equal 0 as \( x \to 0^\pm \). We compute \( \lim_{x \to 0^-} \sin(kx) = \sin(0) = 0 \) and similarly \( \lim_{x \to 0^+} 3x = 0 \) and so the function is continuous independent of \( k \), i.e. \( k \in \mathbb{R} \).

4. (20 pts) Show the equation \( x + 2 \cos(4x) = 0 \) has at least one solution. Explain your work.

   **Solution:** It must be noted that the function \( f(x) = x + 2 \cos(4x) \) is continuous. We can use the IVT to show existence of roots. Note that \( f(-\pi/8) = -\pi/8 < 0 \) and \( f(\pi/8) = \pi/8 > 0 \) and so there must exist a number \( c \in (-\pi/8, -\pi/8) \) so that \( f(c) = 0 \).
5. Consider the function \( f(x) = \frac{2}{3x+3} \).

(a) (14 pts) Find the rate of change of \( f(x) \) at \( x = a \).

(b) (3 pts) Using part (a) find the rate of change of \( f(x) \) at \( x = -1 \).

(c) (3 pts) Using part (a) find the rate of change of \( f(x) \) at \( x = 0 \).

(d) (20 pts) Using the above information find the equation of two different tangent lines that are parallel to the line that goes through the points \((-2, 4)\) and \((-5, 6)\).

Solution:

(a) We have

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\frac{2}{3x+3} - \frac{2}{3a+3}}{x - a} = \lim_{x \to a} \frac{2}{3(x+1)} \frac{x - a}{x - a} = \frac{2}{3} \lim_{x \to a} \frac{a - x}{(x + 1)(a + 1)} = \frac{2}{3} \lim_{x \to a} \frac{-1}{(x + 1)(a + 1)} = \frac{-2}{3(a+1)^2}
\]

(b) The rate of change at \( x = -1 \) does not exist. The one-sided limits of the derivative both are \(-\infty\) so that is an acceptable answer as well.

(c) Plug in 0... we have \( f'(0) = -2/3 \).

(d) The line that goes through the points \((-2, 4)\) and \((-5, 6)\) has slope \( m = \frac{4 - 6}{-2 - (-5)} = \frac{-2}{3} \). We need to set the derivative equal to \( \frac{-2}{3} \) and solve for \( a \),

\[
\frac{-2}{3(a+1)^2} = \frac{-2}{3} \iff (a + 1)^2 = 1 \iff a = 0 \text{ or } a = -2.
\]

We have \( f(0) = \frac{2}{3} \) and \( f(-2) = -\frac{2}{3} \) and so our two tangent lines are

\[ t_1 : \left( y - \frac{2}{3} \right) = -\frac{2}{3}(x - 0), \text{ and } \]

\[ t_2 : \left( y + \frac{2}{3} \right) = -\frac{2}{3}(x + 2) \]

or

\[ t_1 : y = -\frac{2}{3}(x - 1), \text{ and } \]

\[ t_2 : y = -\frac{2}{3}(x + 3) \]

6. The function \( g(x) = x|/x| \) is differentiable at \( x = 0 \). Show this by

(a) (x pts) Define \( g(x) \) as a piecewise function.

(b) (y pts) Using the definition of the derivative consider the left and right hand limits of the difference quotient at 0.
Solution:

(a) 
\[ g(x) = \begin{cases} 
-x^2, & \text{if } x \leq 0 \\
x^2, & \text{if } x \geq 0 
\end{cases} \]

(b) Left hand limit:

\[ \lim_{h \to 0^-} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0^-} \frac{-(h)^2 - 0}{h} = \lim_{h \to 0^-} -h = 0 \]

and the right hand limit:

\[ \lim_{h \to 0^+} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0^+} \frac{(h)^2 - 0}{h} = \lim_{h \to 0^+} h = 0 \]

and so the derivative exists.