APPM 1350  Exam 3 Solutions  Spring 2013

1. (32 points) Evaluate the following expressions.
(a) \( \frac{d}{dx} \int_0^{1/x} (2t^3 - t^2) \, dt \)
(b) \( \int \frac{\cos x}{(1 + 2 \sin x)^2} \, dx \)
(c) \( \int_{-6}^0 \sqrt{36 - x^2} \, dx \)
(d) \( \int_2^{16} 5 \, \frac{5}{3x} \, dx \)

Solution:
(a) Use the Fundamental Theorem of Calculus and the Chain Rule.
\[ \frac{d}{dx} \int_0^{1/x} (2t^3 - t^2) \, dt = \left( \frac{2}{x} \cdot \frac{1}{x^3} \right) - \left( -\frac{1}{x^2} \right) \]
\[ = \frac{2}{x^3} + \frac{1}{x^4} \]

(b) Let \( u = 1 + 2 \sin x \). Then \( du = 2 \cos x \, dx \) and \( \frac{1}{2} \, du = \cos x \, dx \).
\[ \int \frac{\cos x}{(1 + 2 \sin x)^2} \, dx = \frac{1}{2} \left( \int \frac{du}{u^2} = \frac{1}{2} \left( -\frac{1}{u} \right) + C \right) \]
\[ = -\frac{1}{2(1 + 2 \sin x)} + C \]

(c) The integral equals the area of a quarter-circle with radius 6.
\[ \int_{-6}^0 \sqrt{36 - x^2} \, dx = \frac{1}{4} \pi r^2 = \frac{1}{4} \pi (36) = 9\pi \]

(d) \( \int_2^{16} 5 \, \frac{5}{3x} \, dx = \frac{5}{3} \int_2^{16} \frac{dx}{x} = \frac{5}{3} \left[ \ln |x| \right]_2^{16} \)
\[ = \frac{5}{3} (\ln 16 - \ln 2) = \frac{5}{3} (\ln 8) = \frac{5}{3} (3 \ln 2) = 5 \ln 2 \]

2. (14 points) Let \( p(x) = x^3 + 2x^2 \).
(a) Estimate the area under the curve on the interval \([0, 2]\) using \( n \) evenly spaced subintervals and right endpoints. (Find \( R_n \).) Leave your answer unsimplified.

(b) Find the exact area under the curve by evaluating the limit as \( n \to \infty \) of the expression you found in part (a).

(c) Check your answer by calculating \( \int_0^2 p(x) \, dx \) using the Evaluation Theorem.

Solution:
(a) \( R_n = \sum_{i=1}^{n} p(x_i) \Delta x = \sum_{i=1}^{n} \left[ \left( \frac{2i}{n} \right)^3 + 2 \left( \frac{2i}{n} \right)^2 \right] \frac{2}{n} \)

(b) \( A = \lim_{n \to \infty} R_n \)
\[ = \lim_{n \to \infty} \sum_{i=1}^{n} \left[ \left( \frac{2i}{n} \right)^3 + 2 \left( \frac{2i}{n} \right)^2 \right] \frac{2}{n} \]
\[ = \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \left( \frac{8i^3}{n^3} \right) + \frac{2}{n} \sum_{i=1}^{n} \left( \frac{4i^2}{n^2} \right) \]
\[ = \lim_{n \to \infty} \frac{2}{n} \left[ \left( \frac{8}{n^3} \right) \sum_{i=1}^{n} i^3 + 2 \left( \frac{4}{n^2} \right) \sum_{i=1}^{n} i^2 \right] \]
\[ = \lim_{n \to \infty} \frac{2}{n} \left[ \left( \frac{8}{n^3} \right) \left( \frac{n(n+1)}{2} \right)^2 + 2 \left( \frac{4}{n^2} \right) \left( \frac{n(n+1)(2n+1)}{6} \right) \right] \]
\[
\lim_{n \to \infty} \left[ 4 \left( \frac{n}{n} \right) \left( \frac{n+1}{n} \right) \left( \frac{n+1}{n} \right) + \frac{16}{6} \right] \left( \frac{n}{n} \right) \left( \frac{n+1}{n} \right) \left( \frac{2n+1}{n} \right) \right]
\]
\[
= \left[ 4(1 \times 1 \times 1 \times 1) + \frac{16}{6} (1 \times 1 \times 2) \right] \quad \text{(by DOP)}
\]
\[
= 4 + \frac{16}{3} = \frac{28}{3}
\]

(c) \[
\int_0^2 p(x) \, dx = \int_0^2 (x^3 + 2x^2) \, dx
\]
\[
= \left[ \frac{1}{4}x^4 + \frac{2}{3}x^3 \right]_0^2
\]
\[
= \frac{16}{4} + \frac{16}{3} = 4 + \frac{16}{3} = \frac{28}{3}
\]

3. (12 points) A particle is moving along a straight line with velocity \( v(t) = t^2 - t \) (in m/s).

(a) What is the total displacement of the particle over the interval \( 0 \leq t \leq 4 \)?

(b) What is the total distance traveled over the same interval?

Solution:

(a) Total displacement is
\[
\int_0^4 v(t) \, dt = \int_0^4 (t^2 - t) \, dt
\]
\[
= \left[ \frac{1}{3}t^3 - \frac{1}{2}t^2 \right]_0^4
\]
\[
= \frac{64}{3} - \frac{16}{2} = \frac{64}{3} - 8 = \frac{40}{3} \text{ m}.
\]

(b) Total distance traveled is \( \int_0^4 |v(t)| \, dt \). Here \( v(t) = t^2 - t = t(t-1) \) so \( v(t) < 0 \) on \((0, 1)\) and \( v(t) > 0 \) on \((1, \infty)\).

Thus,
\[
\int_0^4 |v(t)| \, dt = \int_0^1 -v(t) \, dt + \int_1^4 v(t) \, dt
\]
\[
= \int_0^1 (t^2 - t) \, dt + \int_1^4 (t^2 - t) \, dt
\]
\[
= \left[ \frac{1}{2}t^2 - \frac{1}{3}t^3 \right]_0^1 + \left[ \frac{1}{3}t^3 - \frac{1}{2}t^2 \right]_1^4
\]
\[
= \frac{1}{2} - \frac{1}{3} + \left[ \frac{1}{3}t^3 - \frac{1}{2}t^2 \right] \bigg|_1^4
\]
\[
= \frac{1}{2} - \frac{1}{3} + \left[ \frac{64}{3} - \frac{16}{2} - \left( \frac{1}{3} \cdot \frac{1}{2} \right) \right]
\]
\[
= \frac{1}{2} + \frac{40}{3} + \frac{1}{2} = \frac{41}{3} \text{ m}.
\]

4. (10 points) Use one iteration of Newton’s Method to approximate \( \sqrt{3} \) starting with an initial guess of \( x_1 = 1 \).

Solution:

\[ x = \sqrt{3} \Rightarrow x^3 = 3 \Rightarrow x^3 - 3 = 0 \] We wish to approximate the root of \( f(x) = x^3 - 3 \). Differentiating yields \( f'(x) = 3x^2 \).

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}
\]
\[
= 1 - \frac{1^3 - 3}{3(1^2)}
\]
\[
= 1 - \frac{-2}{3} = \frac{7}{3}.
\]
5. (10 points) Given that \( f(x) \) is odd, \( \int_0^1 f(2x)dx = 1 \), and \( \int_7^2 f(x)dx = 14 \), find \( \int_{-7}^0 f(x)dx \).

Solution:

- \( \int_7^2 f(x)dx = 14 \Rightarrow \int_2^7 f(x)dx = -14 \)
- \( \int_0^1 f(2x)dx = 1 \). Choosing \( u = 2x \), \( du = 2dx \), \( u(1) = 2 \), \( u(0) = 0 \), we get \( \int_0^1 f(2x)dx = \frac{1}{2} \int_0^2 f(u)du = 1 \Rightarrow \int_0^2 f(x)dx = 2 \).
- \( f \) is odd so \( \int_{-7}^7 f(x)dx = 0 \).

Thus,

\[
0 = \int_{-7}^7 f(x)dx = \int_{-7}^0 f(x)dx + \int_0^2 f(x)dx + \int_2^7 f(x)dx = \int_{-7}^0 f(x)dx + 2 - 14
\]

so \( \int_{-7}^0 f(x)dx = 12 \).

6. (12 points) Let \( f \) be a differentiable, one-to-one function.

(a) Copy the graph of \( f \) and add a sketch of the inverse function \( f^{-1} \).

Solution:

(b) Given

\[
\begin{align*}
f(1) &= -\frac{1}{8} & f'(2) &= -\frac{3}{2} \\
f(2) &= -1 & (f^{-1})'(\frac{1}{8}) &= -\frac{8}{3}
\end{align*}
\]

find the following values.

i. \( f^{-1}(-1) \)
ii. \( f(f^{-1}(8)) \)
iii. \( (f^{-1})'(-1) \)

Solution:

i. Since \( f(2) = -1 \), then \( f^{-1}(-1) = 2 \)
ii. The cancellation equation for inverse functions is \( f(f^{-1}(x)) = x \) so \( f(f^{-1}(8)) = 8 \)
iii. The slope of \( f^{-1} \) at \((-1, 2)\) is the reciprocal of the slope of \( f \) at \((2, -1)\) so \( (f^{-1})'(-1) = 1/f'(2) = -2/3 \).
7. (10 points) Suppose that the function \( f(x) \) has a positive derivative for all \( x \) and that \( f(1) = 0 \). Let

\[
g(x) = \int_0^x f(t) \, dt.
\]

Answer TRUE (if always true) or FALSE (if not always true) for the following statements. No explanation is necessary.

(a) \( g(1) \) is negative.

(b) \( g \) is increasing on \((0, 1)\).

(c) \( g \) has a local maximum at \( x = 1 \).

(d) \( g \) has an inflection point at \( x = 1 \).

(e) The average value of \( g \) on \([0, 1]\) is negative.

Solution:

(a) TRUE. Since \( f \) is an increasing function and \( f(1) = 0 \), then \( f \) is negative on the interval \([0, 1)\). Therefore \( g(1) = \int_0^1 f(t) \, dt \) is negative.

(b) FALSE. Since \( g'(x) = f(x) \) and \( f \) is negative on \((0, 1)\), \( g \) decreases on \((0, 1)\).

(c) FALSE. \( f \) is positive on \((1, \infty)\). Since \( g \) is decreasing on \((0, 1)\) and increasing on \((1, \infty)\), \( g \) has a local minimum at \( x = 1 \).

(d) FALSE. \( g'(x) = f(x) \) and \( g''(x) = f'(x) \). Since \( f' \) is positive for all \( x \), \( g'' \) is also positive so the graph of \( g \) is concave up and does not change concavity.

(e) TRUE. Since \( g \) is negative on \((0, 1)\), \( g_{\text{ave}} = \int_0^1 g(x) \, dx \) is also negative.