1. (15 points)

(a) Find the linearization of \( f(x) = \sqrt{1-x} \) at \( x = 0 \).

(b) Use the linearization to approximate the value of \( \sqrt[4]{0.92} \).

Solution:

(a) First find the values of \( f(0) \) and \( f'(0) \).

\[
\begin{align*}
  f(x) &= (1-x)^{1/4} \\
  f(0) &= 1 \\
  f'(x) &= -\frac{1}{4}(1-x)^{-3/4} = -\frac{1}{4(1-x)^{3/4}} \\
  f'(0) &= -\frac{1}{4}
\end{align*}
\]

Then find \( L(x) \).

\[
L(x) = f(0) + f'(0)(x) = 1 - \frac{1}{4}x
\]

(b) Note that \( \sqrt[4]{0.92} = f(0.08) \approx L(0.08) \).

\[
L(0.08) = 1 - \frac{1}{4}(0.08) = 1 - 0.02 = 0.98
\]

2. (30 points) Consider the function \( f(x) = \frac{-2x}{x^2-3} \). \( f'(x) = \frac{2(x^2+3)}{(x^2-3)^2} \). \( f''(x) = \frac{-4x(x^2+9)}{(x^2-3)^3} \).

(a) Find any vertical, horizontal, or slant asymptotes of \( f \). Use appropriate limits to justify your answer.

(b) On what intervals is \( f \) increasing? decreasing?

(c) Find all local maximum and minimum values of \( f \).

(d) On what intervals is \( f \) concave up? concave down?

(e) Find all inflection points of \( f \).

(f) Using the information from (a) to (e), sketch a graph of \( f \). Clearly label any asymptotes, local extrema, and inflection points.

Solution:

(a) \( f \) is not defined at \( x = \pm \sqrt{3} \) so we check at these x-values for asymptotes.

\[
\begin{align*}
  \lim_{x \to -\sqrt{3}^-} \frac{-2x}{x^2-3} &= \frac{2\sqrt{3}}{0^+} = +\infty, \\
  \lim_{x \to -\sqrt{3}^+} \frac{-2x}{x^2-3} &= \frac{2\sqrt{3}}{0^-} = -\infty, \\
  \lim_{x \to \sqrt{3}^-} \frac{-2x}{x^2-3} &= \frac{-2\sqrt{3}}{0^-} = +\infty, \\
  \lim_{x \to \sqrt{3}^+} \frac{-2x}{x^2-3} &= \frac{-2\sqrt{3}}{0^+} = -\infty.
\end{align*}
\]

So there are vertical asymptotes at \( x = \pm \sqrt{3} \).

\[
\begin{align*}
  \lim_{x \to -\infty} \frac{-2x}{x^2-3} &= \lim_{x \to -\infty} \frac{x^2}{(1-3/x^2)} = -\frac{2}{1} = 0, \\
  \lim_{x \to \infty} \frac{-2x}{x^2-3} &= \lim_{x \to \infty} \frac{2}{(1-3/x^2)} = \frac{2}{1} = 0.
\end{align*}
\]

So there is a horizontal asymptote at \( y = 0 \).

(b) \( f'(x) = \frac{2(x^2+3)}{(x^2-3)^2} > 0 \) so \( f \) is always increasing where defined on \( (-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty) \).
(c) Since $f$ is always increasing there are no local extrema.

(d) $f''(x) = \frac{-4x(x^2 + 9)}{(x^2 - 3)^3}$ so $f'' = 0$ at $x = 0$ and $f''$ is undefined at $x = \pm \sqrt{3}$. Testing values on the four subintervals, we find that $f$ is concave up on $(-\infty, -\sqrt{3} \cup (0, \sqrt{3})$ and concave down on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$.

(e) $f$ switches from concave up to concave down at $x = 0$ and is continuous at $x = 0$, so $(0, f(0)) = (0, 0)$ is the only inflection point.

(f) Horizontal asymptotes $x = -\frac{1}{2}$, $x = 3^{1/2}$, $y = 0$. Vertical asymptote Inflection point $(0,0)$.

3. (15 points) The second hand on a stopwatch, 5 centimeters in length, makes a full revolution every minute. Let $x$ represent the distance between the tip of the hand and its starting position at the 60-second mark. At what rate is $x$ increasing when the hand reaches the 15-second mark? Express your answer in centimeters per second.

**Solution:**

We wish to find $dx/dt$ when $\theta = \pi/2$. The second hand makes a full revolution each minute, or $2\pi$ radians every 60 seconds, so $d\theta/dt = 2\pi/60 = \pi/30$ rad/sec. Use the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$  

$x^2 = 5^2 + 5^2 - 2(5)(5) \cos \theta$  

$x^2 = 50 - 50 \cos \theta$

Differentiate with respect to time.

$$2x \frac{dx}{dt} = 50 \sin \theta \frac{d\theta}{dt}$$  

$$x \frac{dx}{dt} = 25 \sin \theta \frac{d\theta}{dt}$$

Note that when $\theta = \pi/2$, then $x = \sqrt{50} = 5\sqrt{2}$.

$$5\sqrt{2} \frac{dx}{dt} = 25 \left(\sin \frac{\pi}{2}\right) \left(\frac{\pi}{30}\right) = 25 \left(\frac{\pi}{30}\right)$$  

$$5\sqrt{2} \frac{dx}{dt} = \frac{5\pi}{6}$$
\[
\frac{dx}{dt} = \frac{\pi}{6\sqrt{2}} \text{ cm/sec}
\]

4. (12 points) Let \( f(x) = \frac{1}{x} \), where \( 0 < a < b \).

(a) Verify that \( f \) satisfies the hypotheses of the Mean Value Theorem.

(b) Find the value(s) of \( c \) that satisfy the conclusion of the Mean Value Theorem. Express your answer in terms of \( a \) and \( b \).

Solution:

(a) The derivative of \( f \) is \( f'(x) = -\frac{1}{x^2} \). Both \( f \) and \( f' \) are undefined at \( x = 0 \) but we are given that \( a \) and \( b \) are positive so \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), satisfying the hypotheses of the Mean Value Theorem.

(b) The Mean Value Theorem states there there is a \( c \) in \((a, b)\) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

\[
f'(c) = \frac{1}{b - a} \cdot \frac{a - b}{ab} = -\frac{1}{ab}
\]

\[
c = \sqrt{ab}
\]

5. (12 points) For the following statements, answer TRUE if the statement is always true and justify your answer. Otherwise provide a sketch of a COUNTEREXAMPLE to show that the statement may be false.

(a) If \( f \) is differentiable for all \( x \), then \( f \) has an absolute minimum value on \([-5, 5] \).

(b) If \( g \) is decreasing for \( x < -2 \) and increasing for \( x > -2 \), then \( g \) has a local minimum value at \( x = -2 \).

(c) If \( h \) is continuous and \( h(-3) = h(7) \), then there is a number \( c \) in \((-3, 7)\) such that \( h'(c) = 0 \).

6. (16 points) Hank Hill is designing a propane tank with a volume of \( 64\pi \) cubic meters. The tank is cylindrical with spherical endcaps. The spherical endcaps cost \( \frac{8}{3} \) as much per square meter as the cylindrical body. What dimensions will minimize the cost of materials for the tank?
Solution:
We start by writing down the volume and surface area,
\[ V = \frac{4}{3}\pi r^3 + \pi r^2 h = 64\pi, \]
\[ S = 4\pi r^2 + 2\pi rh. \]

Since the spherical endcaps cost eight-thirds as much the sidewalls we write down the cost function to be minimized,
\[ C(r, h) = \frac{32}{3}\pi r^2 + 2\pi rh. \]

To get cost as a function of a single variable we look to the volume equation,
\[ V = \frac{4}{3}\pi r^3 + \pi r^2 h = 64\pi \]
\[ \pi r^2 h = 64\pi - \frac{4}{3}\pi r^3 \]
\[ h = \frac{64}{r^2} - \frac{4}{3}r, \]
and substitute to get
\[ C(r) = \frac{32}{3}\pi r^2 + 2\pi r \left( \frac{64}{r^2} - \frac{4}{3}r \right) \]
\[ = \frac{32}{3}\pi r^2 + \frac{128\pi}{r} - \frac{8}{3}\pi r^2 \]
\[ = 8\pi r^2 + \frac{128\pi}{r}. \]

We check for critical numbers. The domain of \( C(r) \) is \((0, \infty)\) and \( C(r) \) is defined for all \( r > 0 \) so the only critical points we have are those when \( C'(r) = 0 \).
\[ C'(r) = 16\pi r - \frac{128\pi}{r^2} = 0 \]
\[ 16\pi r^3 = 128\pi \]
\[ r^3 = 8 \]
\[ r = 2. \]

So \( r = 2 \) is the only critical number of \( C(r) \).

Now,
\[ C'(r) = 16\pi r - \frac{128\pi}{r^2} > 0 \]
\[ 16\pi r^3 > 128\pi \]
\[ r^3 > 8 \]
\[ r > 2, \]
and
\[ C'(r) = 16\pi r - \frac{128\pi}{r^2} < 0 \]
\[ 16\pi r^3 < 128\pi \]
\[ r^3 < 8 \]
\[ r < 2. \]

So \( r = 2 \) is a critical number of \( C(r) \) and \( C(r) \) is increasing when \( r > 2 \) and \( C(r) \) is decreasing when \( r < 2 \). Thus by the First Derivative Test for Absolute Extrema the cost is minimized when \( r = \frac{2}{3} \) meters and
\[ h = \frac{64}{2^2} - \frac{4}{3}(2) = 16 - \frac{8}{3} = \frac{48 - 8}{3} = \frac{40}{3} \text{ meters}. \]