1. (12 points) Consider the function \( y = f(x) \). Use transformations to match the following functions to the graphs shown. No explanation is necessary.

(a) \( y = f(x + 1) \)  
(b) \( y = f(2x) \) 
(c) \( y = f(-x) - 1 \)  
(d) \( y = |f(x) - 1| \)

Solution:
(a) 8  
(b) 1  
(c) 4  
(d) 2

2. (10 points) Let \( f(x) = \sin x \) and \( g(x) = \frac{x}{x^2 + 2} \).

(a) Find \((g \circ f)(x)\).
(b) What is the domain of \( g \circ f \)?
(c) Is \( g \circ f \) even, odd, or neither? Justify your answer.

Solution:
(a) \((g \circ f)(x) = g(f(x)) = g(\sin x) = \frac{\sin x}{\sin^2 x + 2}\)
(b) Observe that the function \( \sin x \) is defined for all \( x \), and since \( \sin^2 x \geq 0 \), the denominator cannot equal 0. The domain is \( (-\infty, \infty) \).
(c) Check \((g \circ f)(-x)\). Note that \( \sin x \) is an odd function and therefore \( \sin(-x) = -\sin x \).

\[
(g \circ f)(-x) = \frac{\sin(-x)}{\sin^2(-x) + 2} = \frac{-\sin x}{\sin^2 x + 2} = -(g \circ f)(x)
\]
Since \((g \circ f)(-x) = -(g \circ f)(x)\), then \( g \circ f \) is an odd function.

3. (14 points) Let \( f(x) = \sqrt{5 - 4x} \).

(a) Use the definition of the derivative to find \( f'(x) \).
(b) Find an equation of the normal line to the curve \( y = f(x) \) at \( x = -1 \).

Solution:
(a) \( f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \)

\[
= \lim_{h \to 0} \frac{\sqrt{5 - 4(x + h)} - \sqrt{5 - 4x}}{h}
\]
Multiply by the conjugate of the numerator.

\[
= \lim_{h \to 0} \frac{\sqrt{5 - 4x - 4h} - \sqrt{5 - 4x}}{h} \cdot \frac{\sqrt{5 - 4x - 4h} + \sqrt{5 - 4x}}{\sqrt{5 - 4x - 4h} + \sqrt{5 - 4x}}
\]

\[
= \lim_{h \to 0} \frac{5 - 4x - 4h - (5 - 4x)}{h(\sqrt{5 - 4x - 4h} + \sqrt{5 - 4x})}
\]

\[
= \lim_{h \to 0} \frac{-4h}{h(\sqrt{5 - 4x - 4h} + \sqrt{5 - 4x})}
\]

\[
= \lim_{h \to 0} \frac{-4}{\sqrt{5 - 4x - 4h} + \sqrt{5 - 4x}}
\]

\[
= \frac{-4}{2\sqrt{5 - 4x}} = \frac{-2}{\sqrt{5 - 4x}}
\]

(b) The tangent slope is \( f'(-1) = -2/\sqrt{5 + 4} = -2/3 \). The normal slope is the negative reciprocal of the tangent slope, or \( 3/2 \). The point of tangency is \( x = -1, y = f(-1) = 3 \). An equation of the normal line is therefore \( y = 3 + \frac{3}{2}(x + 1) \) or \( y = \frac{3}{2}x + \frac{9}{2} \).

4. (32 points) Evaluate the following limits.

(a) \( \lim_{x \to 3^-} \frac{x^2 + x - 12}{9 - x^2} \)  
(b) \( \lim_{x \to 0^-} \sqrt{\frac{5x^3 - 3|x|}{x}} \)  
(c) \( \lim_{x \to 0^+} \sqrt{x} \cos \frac{\pi}{x} \)
Solution:

(a) \[ \lim_{x \to 3} \frac{x^2 + x - 12}{9 - x^2} = \lim_{x \to 3} \frac{(x + 4)(x - 3)}{3 + x} = \lim_{x \to 3} \frac{x + 4}{3 + x} = \frac{7}{6} \]

(b) Observe that \( \lim_{x \to 0^-} |x| = -x. \)

\[ \lim_{x \to 0^-} \sqrt[3]{\frac{5x^3 - 3|x|}{x}} = \left[ \lim_{x \to 0^-} \frac{5x^3 - 3|x|}{x} \right]^{1/3} = \left[ \lim_{x \to 0^-} \frac{5x^3 + 3x}{x} \right]^{1/3} = \left[ \lim_{x \to 0^-} (5x^2 + 3) \right]^{1/3} = \sqrt[3]{3} \]

(c) Use the Squeeze Theorem. First note that

\[ -1 \leq \cos \frac{\pi}{x} \leq 1 \]
\[ -\sqrt{x} \leq \sqrt{x} \cos \frac{\pi}{x} \leq \sqrt{x}. \]

Next we calculate

\[ \lim_{x \to 0^+} -\sqrt{x} = 0 = \lim_{x \to 0^+} \sqrt{x} \text{ (via direct substitution).} \]

Hence, \( \lim_{x \to 0^+} \sqrt{x} \cos \frac{\pi}{x} = 0 \)

(d) Note that \( \sqrt{x^2} = |x| = -x \) for \( x < 0. \)

\[ \lim_{x \to -\infty} \frac{7x - \sqrt{49x^2 - 8x}}{7x + \sqrt{x^2 - 6x}} = \lim_{x \to -\infty} \frac{7x - \sqrt{x^2 \left( 49 - \frac{8}{x} \right)}}{7x + \sqrt{x^2 \left( 1 - \frac{6}{x} \right)}} = \lim_{x \to -\infty} \frac{7x - |x| \sqrt{49 - \frac{8}{x}}}{7x + |x| \sqrt{1 - \frac{6}{x}}} = \lim_{x \to -\infty} \frac{7 + x \sqrt{49 - \frac{8}{x}}}{7 - x \sqrt{1 - \frac{6}{x}}} = \lim_{x \to -\infty} \frac{7 + \sqrt[3]{49}}{7 - \sqrt[3]{1}} = \frac{14}{6} = \frac{7}{3} \]

5. (10 points) Show that the equation \( \sqrt{x} = \sin x + \frac{1}{2} \) has at least one real root.

Solution:

We wish to show that \( f(x) = \sqrt{x} - \sin x - \frac{1}{2} \) has at least one real root. Since \( \sin x \) is a continuous function and \( \sqrt{x} \) is continuous on \( [0, \infty) \), then \( f \) is continuous on \( [0, \infty) \). We use the Intermediate Value Theorem.

Note that

\[ f(0) = 0 - 0 = 0, \]
\[ f(\pi) = \sqrt{\pi} - 0 = \sqrt{\pi}. \]

Since \( f(0) < \frac{1}{2} \) and \( f(\pi) > \frac{1}{2} \), by the Intermediate Value Theorem, \( f(x) = \frac{1}{2} \) has a solution in the interval \( (0, \pi) \).
6. (12 points) Use the definition of continuity to determine whether the following function \( g \) is continuous at \( x = 0 \).

\[
g(x) = \begin{cases} 
6 \tan(2x) \csc(3x), & x < 0 \\
\sec^4\left(x + \frac{\pi}{4}\right), & x \geq 0 
\end{cases}
\]

**Solution:**
The function \( g \) is continuous at \( x = 0 \) if

\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0).
\]

Evaluate the one-sided limits and find the value of \( g(0) \). We use the theorem \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \).

\[
\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 6 \tan(2x) \csc(3x)
= \lim_{x \to 0^-} 6 \cdot \frac{\sin 2x}{\cos 2x} \cdot \frac{1}{\sin 3x}
= \lim_{x \to 0^-} \frac{6 \cdot \sin 2x}{\cos 2x} \cdot \frac{\sin 3x}{3x}
= \lim_{x \to 0^-} \frac{6 \cdot \sin 2x}{\cos 2x} \cdot \frac{\sin 3x}{3x} \cdot \frac{1 \cdot 2}{1 \cdot 3}
= \frac{6}{1} \cdot \frac{1}{3} \cdot 1 = 4
\]

\[
\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \sec^4\left(x + \frac{\pi}{4}\right)
= \sec^4\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^4 = 4
\]

\[
g(0) = \sec^4\left(\frac{\pi}{4}\right) = 4
\]

Since \( \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0) \), \( g \) is continuous at \( x = 0 \).

7. (10 points) Find a parabola with equation \( y = ax^2 + bx + c \) that has slope 1 at \( x = 6 \), slope \(-3\) at \( x = -2 \) and passes through the point \((0,5)\).

**Solution:**
Since the parabola passes through the point \((0,5)\), we can use the coordinates to solve for constant \( c \).

\[
y(0) = a(0) + b(0) + c = 5
\]

\[
c = 5
\]

Next use \( y' \) to solve for \( a \) and \( b \). We are given that \( y'(6) = 1 \) and \( y'(-2) = -3 \).

\[
y' = 2ax + b
\]

\[
y'(6) = 12a + b = 1
\]

\[
y'(-2) = -4a + b = -3
\]

Now subtract the two equations to eliminate \( b \).

\[
16a = 4
\]

\[
a = \frac{1}{4}
\]

\[
b = 1 - 12 \cdot \frac{1}{4} = -2
\]

The parabola is \( y = \frac{x^2}{4} - 2x + 5 \).