1. The following parts are not related:
   (a) (10 pts) Approximate the area of the region bounded by the curve \( y = x^2 + 4 \) from \( x = -4 \) to \( x = 4 \) and the \( x \)-axis using a Riemann Sum with 4 subintervals of equal length and taking the sample points to be midpoints.

   \[
   \int_{-4}^{4} (x^2 + 4) \, dx \approx 2 \left[ f(-3) + f(-1) + f(1) + f(3) \right] = 2[13 + 5 + 5 + 13] = 2(36) = 72
   \]

   (b) The following limit of Riemann Sums, \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{16i}{n^2} \sqrt{16 - \frac{16i^2}{n^2}} \), describes the area of the region bounded by some function \( f(x) \) for \( 0 \leq x \leq 4 \) and the \( x \)-axis using subintervals of equal length and \( x_i^* = x_i \).

   (i) (6 pts) What is the function \( f(x) \)?

   (ii) (9 pts) What is the area of the region described by the limit? (Hint: Interpret the limit as a definite integral.)

   Solution:

   (a) (10 pts) Note that here \( \Delta x = \frac{4 - (-4)}{4} = 8/4 = 2 \) and so the subintervals are \([-4, -2], [-2, 0], [0, 2], \) and \([2, 4]\), so clearly the midpoints are \( x_1^* = -3, x_2^* = -1, x_3^* = 1, x_4^* = 3 \), thus we have

   \[
   \int_{-4}^{4} (x^2 + 4) \, dx \approx 2 \left[ f(-3) + f(-1) + f(1) + f(3) \right] = 2[13 + 5 + 5 + 13] = 2(36) = 72
   \]

   (b) (i) (6 pts) Here we have \( \Delta x = \frac{4 - 0}{n} = \frac{4}{n} \) and \( x_i^* = x_i = 0 + i \Delta x = \frac{4i}{n} \), so

   \[
   \lim_{n \to \infty} \sum_{i=0}^{n} \frac{16i}{n^2} \sqrt{16 - \frac{16i^2}{n^2}} = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{4i}{n} \sqrt{16 - \left( \frac{4i}{n} \right)^2} \cdot \frac{4}{n} = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i^*) \Delta x = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i^*) \Delta x \text{ so, } f(x) = x \sqrt{16 - x^2}.
   \]

   (b) (ii) (9 pts) We determine the limit by evaluating the definite integral using the substitution \( u = 16 - x^2 \), then \( du = -2xdx \) and,

   \[
   \int_{0}^{4} x \sqrt{16 - x^2} \, dx = -\frac{1}{2} \int_{0}^{6} u^{1/2} \, du = \frac{1}{2} \int_{0}^{16} u^{1/2} \, du = \frac{1}{2} \frac{2}{3} u^{3/2} \bigg|_{0}^{16} = \frac{1}{3} (16)^{3/2} = \frac{64}{3}
   \]

2. The following problems are not related.

   (a) (10 pts) Use Newton’s method to find \( x_2 \), the second approximation of the intersection point of the functions \( y = \sin(x) \) and \( y = \cos(x) \), if the initial approximation is \( x_1 = \frac{\pi}{2} \).

   (b) A square swimming pool with base width \( x \) meters and fixed depth of \( y \) meters is being constructed. The inside walls and floor of the pool are to be painted with a special water-proof paint. There is enough paint to cover exactly 300 \( m^2 \) of surface and the builder plans to use it all up for the painting of this pool:

   (i) (12 pts) What is the largest possible volume of such a pool?

   (ii) (3 pts) How do you know your answer is a maximum? (Justify your answer based on the theories of this class.)

   Solution:

   (a) (10 pts) We wish to approximate a solution to \( \sin(x) = \cos(x) \) which implies \( \sin(x) - \cos(x) = 0 \), so, in Newton’s Method, if we let \( f(x) = \sin(x) - \cos(x) \) then

   \[
   x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{\pi}{2} - \frac{\sin(\pi/2) - \cos(\pi/2)}{\cos(\pi/2) + \sin(\pi/2)} = \frac{\pi}{2} - \frac{1 - 0}{1 + 0} = \frac{\pi}{2} - 1.
   \]
3. The following problems are not related:

(a) (10 pts) Given that \( g(x) \) is an odd function, \( \int_{-2}^{2} g(x) \, dx = 13 \) and \( \int_{-2}^{2} g(x) \, dx = 4 \), find \( \int_{-2}^{5} 3g(x) \, dx \).

(b) Given that \( F(x) = \int_{-2}^{2} \sqrt{5+t^2} \, dt \), answer the following questions without attempting to evaluate any integrals:
   (i) (3 pts) Is \( F(-2) \) positive, negative or neither?
   (ii) (6 pts) On what interval(s) is the function \( F(x) \) increasing? decreasing?
   (iii) (6 pts) Find the linearization of \( F(x) \) at \( x = -1 \).

Solution:

(a) (10 pts) Note that since \( g \) is odd, \( \int_{-2}^{2} g(x) \, dx = 0 \) and so

\[
\int_{-2}^{5} 3g(x) \, dx = 3 \left[ \int_{-2}^{2} g(x) \, dx + \int_{2}^{5} g(x) \, dx \right] = 3 \int_{2}^{5} g(x) \, dx
\]

and now note \( \int_{2}^{5} g(x) \, dx = \int_{2}^{5} g(x) \, dx + \int_{2}^{7} g(x) \, dx \), thus

\[
\int_{2}^{5} g(x) \, dx = \int_{2}^{7} g(x) \, dx - \int_{5}^{7} g(x) \, dx = 13 - 4 = 9
\]

and so, finally, \( \int_{-2}^{5} 3g(x) \, dx = 3 \int_{2}^{5} g(x) \, dx = 3 \cdot 9 = 27 \).

(b)(i) (3 pts) Note that

\[
F(-2) = \int_{-2}^{-4} \sqrt{5+t^2} \, dt - \int_{-4}^{2} \sqrt{5+t^2} \, dt
\]

and, since \( \sqrt{5+t^2} > 0 \) for \(-4 < t < -2\), we have, \( \int_{-4}^{2} \sqrt{5+t^2} \, dt > 0 \) and thus \( F(-2) = - \int_{-4}^{2} \sqrt{5+t^2} \, dt < 0 \) so \( F(-2) \) is negative.

(b)(ii)(6 pts) We first find \( F'(x) \) by applying the Fundamental Theorem of Calculus,

\[
F'(x) = \frac{d}{dx} \left[ \int_{-2}^{2} \sqrt{5+t^2} \, dt \right] = \sqrt{5+(2x)^2} \cdot 2 = 2\sqrt{5+4x^2}
\]

and so \( F'(x) > 0 \) for all \( x \). So, \( F(x) \) is increasing on the interval \((-\infty, \infty)\) and is decreasing for no values of \( x \).
(b)(iii)(6 pts) Note that \( F(-1) = \int_{-2}^{-1} \sqrt{5+1^2} \, dt = 0 \) and \( F'(-1) = 2\sqrt{5+4} = 6 \) (from part (ii)) and so the linearization of \( F(x) \) is

\[
L(x) = L(x) = F(-1) + F'(-1)(x+1) = 6(x+1) = 6x + 6.
\]

4. The following problems are not related.

(a)(i)(15 pts) Evaluate these integrals: (i) \( \int \sin(x) \cot(x) \, dx \) (ii) \( \int_1^\sqrt{2} 2x^3 \sqrt{x^2 - 1} \, dx \) (iii) \( \int_{-2}^{2} \sqrt{16-4x^2} \, dx \)

(b)(10 pts) Show that \( \int_0^1 x^{10}(1-x)^6 \, dx = \int_0^1 x^6(1-x)^{10} \, dx \). Justify your answer.

Solution:

(a)(i)(5 pts) Here we have,

\[
\int \sin(x) \cot(x) \, dx = \int \sin(x) \cdot \frac{\cos(x)}{\sin(x)} \, dx = \int \cos(x) \, dx = \sin(x) + C
\]

(a)(ii)(5 pts) Using the substitution \( u = x^2 - 1 \) we have \( du = 2x \, dx \) and \( x^2 = u + 1 \), thus

\[
\int_1^\sqrt{2} 2x^3 \sqrt{x^2 - 1} \, dx = \int_1^{\sqrt{2}} x^2 \sqrt{x^2 - 1} \, 2x \, dx = \int_0^1 (u + 1) \sqrt{u} \, du = \int_0^1 (u^{3/2} + u^{1/2}) \, du
\]

and so

\[
\int_0^1 (u^{3/2} + u^{1/2}) \, du = \left( \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) \bigg|_{u=0}^{u=1} = \frac{2}{5} + \frac{2}{3} = \frac{6}{15} + \frac{10}{15} = \frac{16}{15}
\]

(a)(iii)(5 pts) First note that

\[
\int_{-2}^{2} \sqrt{16-4x^2} \, dx = \int_{-2}^{2} \sqrt{4(4-x^2)} \, dx = 2 \int_{-2}^{2} \sqrt{4-x^2} \, dx
\]

and the graph of \( y = \sqrt{4-x^2} \) is the top half of the circle centered at the origin with radius \( r = 2 \), so

\[
2 \int_{-2}^{2} \sqrt{4-x^2} \, dx = 2 \cdot \frac{\pi r^2}{2} \bigg|_{r=2} = 4\pi
\]

(b)(10 pts) If we use the substitution \( u = 1-x \), then \( du = -dx \) and \( x = 1-u \), and so

\[
\int_0^1 x^{10}(1-x)^6 \, dx = - \int_1^0 (1-u)^{10} u^6 \, du = \int_0^1 (1-u)^{10} u^6 \, du
\]

and since the variable “\( u \)” is a “dummy variable” in the definite integral we can replace it with “\( x \)” and so we have

\[
\int_0^1 x^{10}(1-x)^6 \, dx = \int_0^1 (1-u)^{10} u^6 \, du = \int_0^1 u^6(1-u)^{10} \, du = \int_0^1 u^6(1-x)^{10} \, dx
\]