1. (8 points) Match the graphs shown to four of the following functions.  
No explanation is necessary.

(a) \( y = -e^x \)  
(b) \( y = e^{-x} \)  
(c) \( y = \ln x \)  
(d) \( y = \log_4 x \)  
(e) \( y = \sin^{-1} x \)  
(f) \( y = \tan^{-1} x \)

**Solution:** (1) e (2) f (3) c (4) d

2. (32 points) Evaluate the following.

(a) Find \( f'(\frac{\sqrt{3}}{2}) \) if \( f(u) = \ln(\frac{3}{\cos^{-1}u}) \). Simplify your answer.

(b) If \( g(x) = Cx^a \), \( g(1) = 3 \), and \( g(3) = \frac{4}{3} \), find the positive constants \( C \) and \( a \).

(c) Find \( \frac{dy}{dx} \) given \( x^2y = (\sqrt{y})^x \), \( x > 0 \), \( y > 0 \). Leave your answer unsimplified.

(d) Find the local maximium and minimum values of \( h(t) = (\ln(t + 5))^2 \), if any.

**Solution:**

(a) 

\[
f(u) = \ln\left(\frac{3}{\cos^{-1}u}\right) = \frac{1}{3} \ln(\cos^{-1}u)
\]

\[
f'(u) = \frac{1}{3} \cdot \frac{1}{\cos^{-1}u} \left(\frac{-1}{\sqrt{1-u^2}}\right)
\]

\[
f'\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{3} \cdot \frac{1}{\pi/6} \left(\frac{-1}{\sqrt{1/4}}\right) = \frac{4}{\pi}
\]

(b) We are given

\( g(1) = Ca = 3 \), \( g(3) = Ca^3 = \frac{4}{3} \).

Divide the two equations.

\[
a^2 = \frac{4}{9} \Rightarrow a = \frac{2}{3}
\]

\[
g(1) = Ca = C\left(\frac{2}{3}\right) = 3 \Rightarrow C = \frac{9}{2}
\]
(c) Use logarithmic and implicit differentiation.

\[ x^{2y} = (\sqrt{y})^x = y^{x/2} \]

\[ 2y \ln x = \frac{x}{2} \ln y \]

\[ 2y \cdot \frac{1}{x} + 2 \ln x \cdot \frac{dy}{dx} = \frac{x}{2} \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \frac{1}{2} \ln y \]

\[ 2 \ln x \cdot \frac{dy}{dx} - x \cdot \frac{dy}{2y} = \frac{1}{2} \ln y - \frac{2y}{x} \]

\[ \frac{dy}{dx} \left( 2 \ln x - \frac{x}{2y} \right) = \frac{1}{2} \ln y - \frac{2y}{x} \]

\[ \frac{dy}{dx} = \frac{\frac{1}{2} \ln y - \frac{2y}{x}}{2 \ln x - \frac{x}{2y}} \]

(d)

\[ h(t) = (\ln(t + 5))^2 \Rightarrow h'(t) = 2 \ln(t + 5) \cdot \frac{1}{t + 5} = \frac{2 \ln(t + 5)}{t + 5} \]

Solve for the critical numbers where \( h' = 0 \). The derivative \( h' \) is defined for all \( t \) in \((-5, \infty)\), the domain of \( h \).

\[ h'(t) = \frac{2 \ln(t + 5)}{t + 5} = 0 \Rightarrow \ln(t + 5) = 0 \Rightarrow t = -4 \]

Use the first derivative test to determine whether there is a maximum or minimum value at the critical number \( t = -4 \). Since \( h'(-\frac{9}{2}) = 4 \ln \left( \frac{1}{2} \right) < 0 \) and \( h'(-3) = \ln 2 > 0 \), there is an absolute minimum value of \( h(-4) = (\ln 1)^2 = 0 \) and no maximum values.

3. (20 points)

(a) \[ \int_e^{e^6} \frac{dx}{x \log_6 x} \]

(b) \[ \int \frac{\sec^2(e^{-3x})}{e^{3x}} \, dx \]

(c) \[ \int \frac{\tan^{-1}(2t)}{1 + 4t^2} \, dt \]

Solution:

(a) First convert to natural log. Then let \( u = \ln x \), \( du = dx/x \). The new limits are then \( u = 1 \) to 6.

\[ \int_e^{e^6} \frac{dx}{x \log_6 x} = \int_e^{6} \frac{\ln 6}{x \ln x} \, dx = (\ln 6) \int_1^6 \frac{du}{u} = (\ln 6) \ln |u| \bigg|_1^6 = (\ln 6)(\ln 6 - \ln 1) = \boxed{(\ln 6)^2} \]

(b) Let \( u = e^{-3x} \), \( du = -3e^{-3x} \, dx \).

\[ \int \frac{\sec^2(e^{-3x})}{e^{3x}} \, dx = -\frac{1}{3} \int \sec^2 u \, du = -\frac{1}{3} \tan u + C = \boxed{-\frac{1}{3} \tan (e^{-3x}) + C} \]

(c) Let \( u = \tan^{-1}(2t) \), \( du = 2 \, dt / (1 + 4t^2) \).

\[ \int \frac{\tan^{-1}(2t)}{1 + 4t^2} \, dt = \frac{1}{2} \int u \, du = \frac{1}{2} \cdot \frac{u^2}{2} + C = \boxed{\frac{1}{4} \left( \tan^{-1}(2t) \right)^2 + C} \]
4. (20 points) Let \( f(x) = \frac{\ln x}{2 + 3 \ln x} \).

(a) Find the domain of \( f \).

(b) Evaluate \( \lim_{x \to 0^+} f(x) \).

(c) Find an equation for the line tangent to \( y = f(x) \) at \( x = \frac{1}{e} \).

(d) Show that \( f \) is one-to-one.

(e) Find the inverse function \( f^{-1} \).

Solution:

(a) The function is undefined if \( x = e^{-2/3} \) or \( x \leq 0 \). The domain of \( f \) is \((0, e^{-2/3}) \cup (e^{-2/3}, \infty)\).

(b) \( \lim_{x \to 0^+} \frac{\ln x}{2 + 3 \ln x} = \lim_{x \to 0^+} \frac{1}{\ln x + 3} = \frac{1}{3} \) since \( \lim_{x \to 0^+} \ln x = -\infty \) and \( \lim_{x \to 0^+} \frac{2}{\ln x} = 0 \).

(c)

\[
f'(x) = \frac{(2 + 3 \ln x)^{\frac{1}{x}} - \ln x \cdot \frac{3}{x}}{(2 + 3 \ln x)^2} = \frac{2 + 3 \ln x - 3 \ln x}{x(2 + 3 \ln x)^2} = \frac{2}{x(2 + 3 \ln x)^2}
\]

\[
f'(\frac{1}{e}) = \frac{2}{e(2 + 3(-1))^2} = 2e
\]

\[
f(\frac{1}{e}) = \frac{-1}{2 + 3(-1)} = 1
\]

The tangent line is \( y = 1 + 2e \left( x - \frac{1}{e} \right) = 2ex - 1 \).

(d) Because \( x > 0 \) and \((2 + 3 \ln x)^2 > 0\), the derivative \( f' \) is always positive. Therefore \( f \) is an increasing function and one-to-one.

(e) Solve for \( x \), then swap \( x \) and \( y \).

\[
y = \frac{\ln x}{2 + 3 \ln x}
\]

\[
2y + 3y \ln x = \ln x
\]

\[
2y = (1 - 3y) \ln x
\]

\[
\ln x = \frac{2y}{1 - 3y}
\]

\[
x = e^{2y/(1-3y)}
\]

\[
f^{-1}(x) = \frac{e^{2x/(1-3x)}}
5. (20 points) Use these approximations to calculate your answers to the following problems:

\[
\begin{align*}
\ln 2 & \approx 0.7, \\
\ln 3 & \approx 1.1, \\
\ln 5 & \approx 1.6, \\
\ln 11 & \approx 2.4.
\end{align*}
\]

(a) How long will it take an investment of \(D\) dollars to quadruple in value if the interest rate is 5% per year compounded continuously?

(b) Superman is locked in a room with a kryptonite-like substance, which renders him powerless. His strength will be restored once the radioactive substance has disintegrated by two-thirds. This will take 22 hours. What is the half-life of the substance?

Solution:

(a) Let \(A(t)\) equal the amount of the investment after \(t\) years. Find \(t\) when \(A(t) = 4D\).

\[
A(t) = De^{0.05t} = 4D
\]

\[
e^{0.05t} = 4
\]

\[
0.05t = \ln 4
\]

\[
t = \frac{\ln 4}{0.05} = \frac{2\ln 2}{0.05} \approx \frac{2(0.7)}{0.05} = \frac{140}{5} = 28 \text{ years}
\]

(b) Let \(m(t)\) equal the amount of kryptonite after \(t\) hours. We are given that \(m(22) = \frac{1}{3}m(0)\). First find the rate constant \(k\).

\[
m(t) = m(0)e^{kt}
\]

\[
m(22) = m(0)e^{22k} = \frac{1}{3}m(0)
\]

\[
e^{22k} = \frac{1}{3}
\]

\[
22k = \ln(1/3)
\]

\[
k = -\frac{\ln 3}{22}
\]

Now find \(t\) when \(m(t) = \frac{1}{2}m(0)\).

\[
m(t) = m(0)e^{kt} = \frac{1}{2}m(0)
\]

\[
e^{kt} = \frac{1}{2}
\]

\[
kt = \ln(1/2)
\]

\[
t = -\frac{\ln 2}{k} = -\ln 2 \left(\frac{-22}{\ln 3}\right)
\]

\[
\approx (0.7) \cdot \frac{22}{1.1} = 7(22)\frac{11}{11} = 14 \text{ hours}
\]