1. (10 points) Find the 100th derivative of the following functions.

(a) \( f(t) = 7^t \) 
(b) \( h(x) = \ln(13x) + \ln \left( \frac{1}{13x} \right) \)

Solution:

(a) \( f(t) = 7^t \), \( f'(t) = 7^t \ln(7) \), \( f''(t) = 7^t \ln(7)^2 \), \( f'''(t) = 7^t \ln(7)^3 \), \ldots, \( f^{(100)}(t) = 7^t \ln(7)^{100} \)

(b) First simplify the function.
\[
h(x) = \ln(13x) + \ln \left( \frac{1}{13x} \right) = (\ln 13 + \ln x) + (\ln 1 - \ln 13 - \ln x) = \ln 1 = 0
\]
\[
h^{(100)}(x) = 0
\]

2. (16 points) For each of the following curves, find an equation of the tangent line at the given value.

(a) \( y = x \ln x \), \( x = e \) 
(b) \( y = (\tan x)^{\ln x} \), \( x = \frac{\pi}{4} \)

Solution:

(a) The \( y \)-coordinate at \( x = e \) is \( y(e) = e \ln e = e \). The tangent slope is
\[
y = x \ln x \\
y' = x \cdot \frac{1}{x} + \ln x = 1 + \ln x \\
y'(e) = 1 + \ln e = 2
\]
The equation of the tangent line is \( y = e + 2(x - e) = 2x - e \). 

(b) The \( y \)-coordinate is \( y(\pi/4) = (\tan(\pi/4))^{\ln(\pi/4)} = 1^{\ln(\pi/4)} = 1 \). The tangent slope can be found using logarithmic differentiation:
\[
y = (\tan x)^{\ln x} \\
\ln y = \ln(\tan x)^{\ln x} = (\ln x) \ln(\tan x) \\
\frac{1}{y} \cdot \frac{dy}{dx} = \ln x \cdot \frac{\sec^2 x}{\tan x} + \ln(\tan x) \cdot \frac{1}{x} \\
\frac{dy}{dx} = (\tan x)^{\ln x} \left( \ln x \cdot \frac{\sec^2 x}{\tan x} + \frac{\ln(\tan x)}{x} \right) \\
\frac{dy}{dx} \bigg|_{x=\pi/4} = (1) \left( \ln \left( \frac{\pi}{4} \right) \cdot \frac{2}{1} + \ln(1) \cdot \frac{4}{\pi} \right) = 2 \ln \left( \frac{\pi}{4} \right)
\]
The equation of the tangent line is \( y = 1 + 2 \ln \left( \frac{\pi}{4} \right) \left( x - \frac{\pi}{4} \right) \).
3. (22 points) Find the values of $x$ where the following curves have horizontal tangents.

(a) $y = x^2 \ln \left( \frac{2}{x} \right)$

(b) $y = \frac{\log_6 x}{x} - 6$

(c) $y = (\sin x)e^{\sin x}, \; 0 \leq x \leq 2\pi$

Solution:

(a)

$y = x^2 \ln \left( \frac{2}{x} \right)$

$y' = x^2 \cdot \frac{x}{2} \left( -\frac{2}{x^2} \right) + 2x \ln(2/x) = -x + 2x \ln(2/x)$

Solve $y' = 0$.

$0 = -x + 2x \ln(2/x)$

$0 = x(-1 + 2 \ln(2/x))$

$x = 0$ is not a solution because $x$ is not in the domain of the function. Here is the only solution:

$0 = -1 + 2 \ln(2/x)$

$\frac{1}{2} = \ln(2/x)$

$e^{1/2} = \frac{2}{x}$

$x = \sqrt{2e} = \frac{2}{\sqrt{e}}$

(b)

$y = \frac{\log_6 x}{x} - 6 = \frac{\ln x}{x \ln 6} - 6$

$y' = \frac{1}{\ln 6} \cdot \frac{x \cdot \frac{1}{x} - \ln x}{x^2}$

$= \frac{1 - \ln x}{x^2 \ln 6}$

Solve $y' = 0$.

$0 = 1 - \ln x$

$x = e$

(c)

$y = (\sin x)e^{\sin x}$

$y' = \sin x(\cos x)e^{\sin x} + (\cos x)e^{\sin x}$
Solve \( y' = 0 \).

\[
0 = (\cos x)e^{\sin x}(\sin x + 1)
\]

Since \( e^{\sin x} > 0 \), either \( \cos x = 0 \) or \( \sin x = -1 \) in \([0, 2\pi]\).

\[
x = \frac{\pi}{2}, \frac{3\pi}{2}
\]

4. (10 points) Let \( y = \frac{e^x}{e^x + 1} \).

(a) Show that \( y \) is an increasing function and therefore one-to-one.

(b) Find the inverse function of \( y \).

Solution:

(a) Show that \( y' > 0 \).

\[
y = \frac{e^x}{e^x + 1} \Rightarrow y' = \frac{(e^x + 1)e^x - e^x \cdot e^x}{(e^x + 1)^2} = \frac{e^x + e^x - e^{2x}}{(e^x + 1)^2} = \frac{e^x}{(e^x + 1)^2}
\]

Since \( y' > 0 \) for all \( x \), \( y \) is an increasing function and one-to-one.

(b) Solve for \( x \), then swap \( x \) and \( y \).

\[
y = \frac{e^x}{e^x + 1} \\
e^x y + y = e^x \\
e^x y - e^x = -y \\
e^x = \frac{-y}{y - 1} = \frac{y}{1 - y} \\
x = \ln \left( \frac{y}{1 - y} \right)
\]

\[
y^{-1} = \ln \left( \frac{x}{1 - x} \right)
\]

5. (10 points) Evaluate the following limits.

(a) \( \lim_{u \to 0} \tan^{-1} \left( e^u \right) \)

(b) \( \lim_{t \to 1^+} \cos^{-1}(\ln t) \)

(c) \( \lim_{x \to \infty} \sin^{-1} \left( \frac{1 - x^2}{1 + 2x^2} \right) \)

Solution:

(a) \( \lim_{u \to 0} \tan^{-1} \left( e^u \right) = \tan^{-1} \left( \lim_{u \to 0} e^u \right) = \tan^{-1}(1) = \frac{\pi}{4} \)

(b) \( \lim_{t \to 1^+} \cos^{-1}(\ln t) = \cos^{-1} \left( \lim_{t \to 1^+} \ln t \right) = \cos^{-1}(0) = \frac{\pi}{2} \)

(c) \( \lim_{x \to \infty} \sin^{-1} \left( \frac{1 - x^2}{1 + 2x^2} \right) = \sin^{-1} \left( \lim_{x \to \infty} \frac{1 - x^2}{1 + 2x^2} \right) = \sin^{-1} \left( -\frac{1}{2} \right) = \frac{-\pi}{6} \)
6. (20 points) Evaluate the following integrals.

(a) \[ \int \tan(3x) \, dx \]
(b) \[ \int \frac{e^{3x}}{1 + e^{6x}} \, dx \]
(c) \[ \int_1^e \frac{\sin(\ln x)}{x} \, dx \]

Solution:

(a) Let \( u = \cos(3x) \), \( du = -3\sin(3x) \, dx \Rightarrow -du/3 = \sin(3x) \, dx \).

\[
\int \tan(3x) \, dx = \int \frac{\sin(3x)}{\cos(3x)} \, dx = -\frac{1}{3} \int \frac{du}{u} = -\frac{1}{3} \ln |u| + C = -\frac{1}{3} \ln |\cos(3x)| + C
\]

(b) Let \( u = e^{3x} \), \( du = 3e^{3x} \, dx \Rightarrow du/3 = e^{3x} \, dx \).

\[
\int \frac{e^{3x}}{1 + e^{6x}} \, dx = \frac{1}{3} \int \frac{du}{1 + u^2} = \frac{1}{3} \arctan u + C = \frac{1}{3} \arctan (e^{3x}) + C
\]

(c) Let \( u = \ln x \), \( du = dx/x \). Change limits to \( u(1) = 0 \), \( u(e^\pi) = \pi \).

\[
\int_1^e \frac{\sin(\ln x)}{x} \, dx = \int_0^\pi \sin u \, du = -\cos u \bigg|_0^\pi = -\cos \pi + \cos 0 = 1 + 1 = 2
\]

7. (12 points) Alvin mistakenly left his smartphone outside overnight in the Alaskan cold. The phone had a temperature of \( -30^\circ C \) when he brought it inside in the morning. Alvin wants to check his messages but the phone won’t work properly until it warms up to \( 0^\circ C \). If the phone reaches a temperature of \( -25^\circ C \) after 2 minutes in Alvin’s \( 20^\circ C \) house, how long will it take before Alvin can use his phone?

(Use the approximations \( \ln 2 \approx 0.7 \), \( \ln 3 \approx 1.1 \), and \( \ln 5 \approx 1.6 \) to calculate your answer.)

Solution: We are given \( T_0 = -30 \), \( T_S = 20 \), \( T(2) = -25 \). We wish to find \( t \) when \( T = 0 \). First find \( k \).

\[
T - T_S = (T_0 - T_S)e^{kt}
\]

\[
-25 - 20 = (-30 - 20)e^{k(2)}
\]

\[
-45 = -50e^{2k}
\]

\[
\frac{9}{10} = e^{2k}
\]

\[
2k = \ln(9/10)
\]

\[
k = \frac{1}{2} \ln(9/10)
\]

Now solve for \( t \) when \( T = 0 \).

\[
0 - 20 = (-30 - 20)e^{kt}
\]

\[
\frac{2}{5} = e^{kt}
\]

\[
k t = \ln(2/5)
\]

\[
t = \frac{\ln(2/5)}{k} = \frac{2 \ln(2/5)}{\ln(9/10)} = \frac{2(\ln 2 - \ln 5)}{\ln 9 - \ln 10} = \frac{2(\ln 2 - \ln 5)}{2 \ln 3 - \ln 2 - \ln 5}
\]

\[
\approx \frac{2(0.7 - 1.6)}{2(1.1) - 0.7 - 1.6} = \frac{2(-0.9)}{2.2 - 2.3} = 18 \text{ min}
\]

The actual value of \( t \) is about 17.4 min.