1. (12 points) Match the graphs shown to the following functions. No explanation is necessary.

(a) \( y = e^x \)  
(b) \( y = e^{-x} \)  
(c) \( y = \ln x \)  
(d) \( y = 4^x \)  
(e) \( y = 4^{-x} \)  
(f) \( y = \log_4 x \)

Solution:

(a) 4  (b) 1  (c) 5  (d) 3  (e) 2  (f) 6

2. (18 points) Find \( \frac{dy}{dx} \) for each of the following equations.

(a) \( y = 5^{\ln x} \)  
(b) \( y = x^{\sqrt{x}} \)  
(c) \( \sec(xy) = \ln(\cos x) \)

Solution:

(a) \[
y = 5^{\ln x} \\
y' = 5^{\ln x} \cdot \frac{1}{x} = \frac{5^{\ln x} \cdot \ln 5}{x}
\]

(b) Solution 1: Use logarithmic differentiation.

\[
y = x^{\sqrt{x}} \\
\ln y = \ln x^{\sqrt{x}} \\
\ln y = \sqrt{x} \ln x
\]

Differentiate using the product rule.

\[
\frac{1}{y} \frac{dy}{dx} = \sqrt{x} \cdot \frac{1}{x} + \ln x \cdot \frac{1}{2\sqrt{x}} \\
\frac{dy}{dx} = x^{\sqrt{x}} \left( \frac{1}{\sqrt{x}} + \ln x \cdot \frac{1}{2\sqrt{x}} \right) = x^{\sqrt{x}} \left( \frac{2 + \ln x}{2\sqrt{x}} \right)
\]

Solution 2: Express \( x \) as \( e^{\ln x} \).

\[
y = x^{\sqrt{x}} = \left( e^{\ln x} \right)^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}} \\
y' = e^{\ln x^{\sqrt{x}}} \cdot \frac{d}{dx} \left( \sqrt{x} \ln x \right) \\
= e^{\ln x^{\sqrt{x}}} \left( \sqrt{x} \cdot \frac{1}{x} + \ln x \cdot \frac{1}{2\sqrt{x}} \right) \\
= x^{\sqrt{x}} \left( \frac{2 + \ln x}{2\sqrt{x}} \right)
\]

(c) \[\sec(xy) = \ln(\cos x)\]

\[
\sec(xy) \tan(xy) (xy' + y) = \frac{1}{\cos x} (-\sin x) \\
\sec(xy) \tan(xy) (xy' + y) = -\tan x
\]

\[
x y' + y = \frac{-\tan x}{\sec(xy) \tan(xy)} \\
x y' = \frac{-\tan x}{\sec(xy) \tan(xy)} - y \\
y' = \frac{-\tan x}{x \sec(xy) \tan(xy)} - \frac{y}{x}
\]
3. (21 points) Evaluate the following integrals. Simplify your answers using logarithm rules, if applicable.

(a) \[ \int \frac{\cos x}{1 + \sin^2 x} \, dx \]
(b) \[ \int_2^5 \frac{5}{3t} \, dt \]
(c) \[ \int_{\ln 2}^{\ln 5} \frac{e^x}{\sqrt{e^x - 1}} \, dx \]

Solution:
(a) Let \( u = \sin x \). Then \( du = \cos x \, dx \).

\[ \int \frac{\cos x}{1 + \sin^2 x} \, dx = \int \frac{du}{1 + u^2} = \tan^{-1} u + C \]
\[ = \tan^{-1} (\sin x) + C \]

(b) \[ \int_2^5 \frac{5}{3x} \, dx = \frac{5}{3} \int_2^5 \frac{dx}{x} = \frac{5}{3} \left[ \ln |x| \right]_2^5 \]
\[ = \frac{5}{3} (\ln 5 - \ln 2) = \frac{5}{3} (\ln 5) = \frac{5}{3} (3 \ln 2) = 5 \ln 2 \]

(c) Let \( u = e^x - 1 \). Then \( du = e^x \, dx \). The new \( u \)-limits are 1 and 4.

\[ \int_{\ln 2}^{\ln 5} \frac{e^x}{\sqrt{e^x - 1}} \, dx = \int_1^4 \frac{1}{\sqrt{u}} \, du = \int_1^4 u^{-1/2} \, du \]
\[ = 2u^{1/2} \bigg|_1^4 = 2(\sqrt{4} - \sqrt{1}) = 2 \]

4. (10 points) Find the local maximum and minimum values of \( f(x) = \frac{e^{x^2}}{x} \).

Critical numbers:
\[ f'(x) = \frac{2x^2 e^{x^2} - e^{x^2}}{x^2} = e^{x^2} \left( 2 - \frac{1}{x^2} \right) \]

- \( f' \) does not exist when \( x = 0 \) but \( f(0) \) also does not exist so there is no extremum at \( x = 0 \).
- \( f'(x) = 0 \) only when \( 2 - \frac{1}{x^2} = 0 \Rightarrow \frac{1}{x^2} = 2 \Rightarrow x^2 = \frac{1}{2} \)
\[ \Rightarrow x = \pm \frac{1}{\sqrt{2}} \]

Local extrema:
- \( f \) is increasing when \( 2 - \frac{1}{x^2} > 0 \Rightarrow x^2 > \frac{1}{2} \Rightarrow |x| > \frac{1}{\sqrt{2}} \).
- Similarly, \( f \) is decreasing when \( |x| < \frac{1}{\sqrt{2}} \).
- So by the first derivative test there is a local maximum at \((-\frac{1}{\sqrt{2}}, -\sqrt{2e})\) and a local minimum at \((\frac{1}{\sqrt{2}}, \sqrt{2e})\).

Plot of \( f(x) = \frac{e^{x^2}}{x} \).
A local maximum is seen at \((-\frac{1}{\sqrt{2}}, -\sqrt{2e})\)
and a local minimum is at \((\frac{1}{\sqrt{2}}, \sqrt{2e})\).
5. (10 points) Let \( y = \frac{\sqrt{1 + x} \sqrt[3]{1 - x}}{(7 - 7x)} \). Find \( y' \) using logarithmic differentiation.

**Solution:**

\[
y = \frac{\sqrt{1 + x} \sqrt[3]{1 - x}}{(7 - 7x)}
\]

\[
\ln y = \frac{1}{2} \ln(1 + x) + \frac{1}{3} \ln |1 - x^3| - \ln |7 - 7x|
\]

\[
\frac{1}{y} y' = \left( \frac{1}{2} \right) \frac{1}{1 + x} + \left( \frac{1}{3} \right) \frac{-3x^2}{1 - x^3} - \frac{-7}{7 - 7x}
\]

\[
\frac{1}{y} y' = \frac{1}{2 + 2x} - \frac{x^2}{1 - x^3} + \frac{1}{1 - x}
\]

\[
y' = \frac{\sqrt{1 + x} \sqrt[3]{1 - x}}{(7 - 7x)} \left[ \frac{1}{2 + 2x} - \frac{x^2}{1 - x^3} + \frac{1}{1 - x} \right].
\]

6. (14 points) For each of the following curves, find an equation of the tangent line at the given value.

(a) \( y = \cos^{-1}(2x), \ x = 0 \)

(b) \( y = \tan^{-1} \left( \frac{x}{4} \right), \ x = -4 \)

**Solution:**

(a) Find \( y(0) \) and \( y'(0) \).

\[
y = \cos^{-1}(2x)
\]

\[
y(0) = \cos^{-1}(0) = \pi/2
\]

\[
y' = \frac{-1}{\sqrt{1 - (2x)^2}} (2)
\]

\[
y(0) = -2
\]

\[
y'(0) = -2
\]

An equation of the tangent line is \( y = \frac{\pi}{2} - 2x \).

(b) Find \( y(-4) \) and \( y'(-4) \).

\[
y = \tan^{-1} \left( \frac{x}{4} \right)
\]

\[
y(-4) = \tan^{-1}(-1) = -\pi/4
\]

\[
y' = \frac{1}{1 + \left( \frac{x}{4} \right)^2} \cdot \frac{1}{4}
\]

\[
y'(-4) = \frac{1}{1 + 1/4} \cdot \frac{1}{4} = \frac{1}{8}
\]

An equation of the tangent line is \( y = -\frac{\pi}{4} + \frac{1}{8}(x + 4) \).

7. (15 points) Whenever Pinocchio tells a lie, his nose lengthens until he cannot turn around in a room. Suppose his nose grows at a rate proportional to its length, measuring 5 cm after one second and measuring 6 cm after three seconds.

(a) Find the relative rate of growth.

(b) What is the normal length of Pinocchio’s nose?

(Use the approximations \( \sqrt{6/5} \approx 1.1 \) and \( (6/5)^{1/3} \approx 1.3 \) to calculate your answer to part (b).)

**Solution:**

(a) We wish to find the rate constant \( k \). We are given that \( y(1) = 5 \) cm and \( y(3) = 6 \) cm.

\[
y(t) = y_0 e^{kt}
\]

\[
y(1) = y_0 e^k = 5
\]

\[
y(3) = y_0 e^{3k} = 6
\]
Divide the two equations.

\[
\frac{e^{3k}}{e^k} = \frac{6}{5} \Rightarrow e^{2k} = \frac{6}{5}
\]

\[2k = \ln(6/5) \Rightarrow k = \frac{\ln(6/5)}{2} \approx 0.09\]

(b) Find \(y_0\).

\[y_0e^k = 5\]
\[y_0e^{\ln(6/5)/2} = 5 \Rightarrow y_0(6/5)^{1/2} = 5\]
\[y_0 = \frac{5}{\sqrt{6/5}} \approx \frac{5}{1.1} = \frac{50}{11} = \frac{6}{11} \text{ cm}\]

Alternative solution:

\[y_0e^{3k} = 6\]
\[y_0e^{3\ln(6/5)/2} = 6 \Rightarrow y_0(6/5)^{3/2} = 6\]
\[y_0 = \frac{6}{\sqrt{(6/5)^3}} \approx \frac{6}{1.3} = \frac{60}{13} = \frac{8}{13} \text{ cm}\]