1. (10 points) Let $f(x) = 3 + x + e^x$.

(a) Find an equation of the tangent line to the curve $y = f(x)$ when $x = 0$.

(b) Is $f$ one-to-one? Why or why not? [Hint: analyze the sign of the derivative]

(c) *Without attempting to find the inverse function*, calculate $(f^{-1})'(4)$.

**SOLUTION**

(a) First we find the derivative: $f'(x) = 1 + e^x$.

So the slope of the tangent line is $f'(0) = 1 + e^0 = 1 + 1 = 2$.

The $y$-coordinate when $x = 0$ is given by $f(0) = 3 + 0 + e^0 = 3 + 1 = 4$,

so using point-slope form an equation of the tangent line is

$$y - 4 = 2(x - 0) \quad \text{or} \quad y = 2x + 4.$$

(b) Yes, $f$ is one-to-one.

From part (a) we know that $f'(x) = 1 + e^x$, which is always positive since $e^x > 0$ for all real $x$.

Hence $f$ is increasing and so one-to-one.

(c) We use the formula

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(f^{-1}(a))}$$

with $a = 4$.

But $f(0) = 4$ from part (a), hence $f^{-1}(4) = 0$.

Moreover, $f'(0) = 2$, also from part (a).

Thus

$$\left(f^{-1}\right)'(4) = \frac{1}{f'(f^{-1}(4))} \quad \text{formula}$$

$$= \frac{1}{f'(0)}$$

$$= \frac{1}{2}.$$
2. (22 points)

(a) What is the definition of \( \ln(x) \)? Be sure to include the domain in your answer.
(b) Sketch a graph and shade in a region whose area is equal to \( \ln(3) \).
(c) Use a left-hand Riemann sum with \( n = 2 \) to approximate \( \ln(3) \). \textit{Simplify your answer.} Is this an overestimate or an underestimate? Why? Sketch a new graph illustrating what you’ve done.
(d) Use a right-hand Riemann sum with \( n = 2 \) to approximate \( \ln(3) \). \textit{Simplify your answer.} Is this an overestimate or an underestimate? Why? Draw yet another new graph illustrating what you’ve done.
(e) Using your answers from parts (c) and (d), copy the following inequality into your bluebook and fill in the blanks:

\[ \underline{\text{_____}} \leq \ln(3) \leq \overline{\text{_____}} \]

\[ \text{SOLUTION} \]

(a) For \( x > 0 \), \( \ln(x) = \int_{1}^{x} \frac{1}{t} \, dt \).

(b) The graph should look something like this:

\[ \text{the curve is } y = \frac{1}{t} \]

Set \( f(t) = \frac{1}{t} \).

(c) With \( n = 2 \), the interval \([1, 3]\) is split into \([1, 2]\) and \([2, 3]\).

The left endpoints are 1 and 2, and \( \Delta t = \frac{3-1}{2} = \frac{2}{2} = 1 \).

Hence

\[ L_2 = f(1)(1) + f(2)(1) = 1 \cdot 1 + \frac{1}{2} \cdot 1 = 1 + \frac{1}{2} = \frac{3}{2}. \]

This is an overestimate since \( f(t) = \frac{1}{t} \) is always decreasing (its derivative is always negative).

The new graph should look something like this:
(d) The right endpoints are 2 and 3, so with $\Delta t = 1$ again the righthand Riemann sum is

$$R_2 = f(2) \cdot 1 + f(3) \cdot 1 = \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 = \frac{5}{6}.$$ 

This is an underestimate since $f(t) = \frac{1}{t}$ is always decreasing.

The new graph should look something like this:

(e) It follows from (c) and (d) that

$$\frac{5}{6} \leq \ln(3) \leq \frac{3}{2}.$$
3. (22 Points) Given

\[ y = \frac{(2 - 3x)^5 \sin^4(x)}{\sqrt{(x^2 + 1)^3}} \]

find \( \frac{dy}{dx} \) using logarithmic differentiation.

**SOLUTION**

First take the absolute value of both sides:

\[ |y| = \left| \frac{(2 - 3x)^5 \sin^4(x)}{\sqrt{(x^2 + 1)^3}} \right| = \frac{|2 - 3x|^5 \sin^4(x)}{\sqrt{(x^2 + 1)^3}} \]

by properties of absolute value and since \( \sin^4(x) \) and \( \sqrt{(x^2 + 1)^3} \) are both nonnegative.

Now take the natural log of both sides and use log properties to rewrite:

\[ \ln(|y|) = \ln \left( \frac{|2 - 3x|^5 \sin^4(x)}{\sqrt{(x^2 + 1)^3}} \right) \]
\[ = \ln(|2 - 3x|^5 \sin^4(x)) - \ln \left( \sqrt{(x^2 + 1)^3} \right) \]
\[ = \ln(|2 - 3x|^5) + \ln(\sin^4(x)) - \ln \left( (x^2 + 1)^{3/2} \right) \]
\[ = 5 \ln(|2 - 3x|) + 4 \ln(\sin(x)) - \frac{3}{2} \ln(x^2 + 1) \]

since

\[ \sin^4(x) = (\sin(x))^4 \quad \text{and} \quad (x^2 + 1)^{3/2} = (x^2 + 1)^{3/2}. \]

Now we differentiate implicitly:

\[ \frac{d}{dx} [\ln(|y|)] = 5 \cdot \frac{d}{dx} \left[ 5 \ln(|2 - 3x|) + 4 \ln(\sin(x)) - \frac{3}{2} \ln(x^2 + 1) \right] \]
\[ = 5 \cdot \frac{d}{dx} [\ln(|2 - 3x|)] + 4 \cdot \frac{d}{dx} [\ln(\sin(x))] - \frac{3}{2} \cdot \frac{d}{dx} [\ln(x^2 + 1)] \]
\[ = 5 \cdot \frac{1}{2 - 3x} \cdot (-3) + 4 \cdot \frac{1}{\sin(x)} \cdot \cos(x) - \frac{3}{2} \cdot \frac{1}{x^2 + 1} \cdot 2x \]

where we used the derivative formula

\[ \frac{d}{dx} [\ln(|u|)] = \frac{1}{u} \cdot u', \]

valid for all nonzero differentiable functions \( u = u(x) \).
Multiplying both sides by $y$, we obtain

$$\frac{dy}{dx} = y \left( \frac{-15}{2 - 3x} + 4 \cot(x) - \frac{3x}{x^2 + 1} \right)$$

$$= \frac{(2 - 3x)^5 \sin^4(x)}{\sqrt{(x^2 + 1)^3}} \left( \frac{-15}{2 - 3x} + 4 \cot(x) - \frac{3x}{x^2 + 1} \right)$$

and we’re done.
4. (24 points) A cup of tea in a room at 60°F cools from 200°F to 120°F in half an hour.

(a) Find an expression for $T(t)$, the temperature of the tea (in °F) after $t$ minutes.
(b) Find the time at which the tea is 90°.
(c) What is the rate of cooling of the tea after 30 minutes?
(d) What is $\lim_{t \to \infty} T(t)$?

Note: You do not have to simplify any answers or constants involved in this problem, but you should include units wherever appropriate.

OVER
(a) We use Newton’s law of cooling, which says that $\frac{dT}{dt} = k(T - T_s)$, where $T_s$ is the surrounding temperature.

Here $T_s = 60$, $T(0) = 200$ and $T(30) = 120$.

Set $y(t) = T(t) - 60$.

Since

$$\frac{dy}{dt} = \frac{dT}{dt} - 0 = \frac{dT}{dt} = k(T - 60) = ky,$$

the function $y$ must have the form $y(t) = y(0)e^{kt}$ for $k$ constant.

But $y(0) = T(0) - 60 = 200 - 60 = 140$, so $y(t) = 140e^{kt}$.

Moreover, $y(30) = T(30) - 60 = 120 - 60 = 60$, and we use this data to find $k$:

$$60 = y(30) = 140e^{k\cdot30} \implies \frac{60}{140} = e^{30k} \implies \ln\left(\frac{3}{7}\right) = 30k \implies k = \frac{1}{30}\ln\left(\frac{3}{7}\right).$$

Hence $y(t) = 140e^{\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)t}$ so that

$$T(t) = y(t) + 60 = 140e^{\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)t} + 60.$$

(b) We need to solve $T(t) = 90$ for $t$:

$$90 = 140e^{\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)t} + 60 \iff 30 = 140e^{\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)t}$$

$$\iff \frac{30}{140} = e^{\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)t}$$

$$\iff \ln\left(\frac{3}{14}\right) = \left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)t$$

$$\iff t = \frac{-\ln\left(\frac{3}{14}\right)}{\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)} \text{ years}$$

(c) The rate of cooling of the tea after 30 minutes is $\frac{dT}{dt} \bigg|_{t=30}$. But

$$\frac{dT}{dt} \bigg|_{t=30} = \frac{dy}{dt} \bigg|_{t=30} = ky(30) = \left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right) \cdot 60 = 2\ln\left(\frac{3}{7}\right)^{\circ}/\text{min}.$$

(d) Note that

$$\lim_{t \to \infty} T(t) = \lim_{t \to \infty} \left(140e^{\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)t} + 60\right) = 0 + 60 = 60^\circ$$

because $\left(\frac{1}{30}\right)\ln\left(\frac{3}{7}\right)$ is negative since $3/7 < 1$ and hence $\ln\left(\frac{3}{7}\right) < 0$.

We also used that $\lim_{t \to \infty} e^{At} = 0$ for any negative constant $A$.

This answer “makes sense” since the tea should approach the temperature of the surrounding room. But that interpretation comes after you calculate! It’s not the reason you get 60°!
5. (22 POINTS) Evaluate the following integrals:

(a) \( \int_{1}^{e^3-1} \frac{1}{t+1} \, dt \). Simplify your answer.

(b) \( \int x^{-2} e^{\pi/x+1} \, dx \)

(c) \( \int_{0}^{1-e^{-2}} \frac{\ln(1-x)}{1-x} \, dx \). Simplify your answer.

**SOLUTION**

(a) Set \( u = t + 1 \) so that \( du = dt \).

When \( t = 1 \) we have \( u = 1 + 1 = 2 \), and when \( t = e^3 - 1 \), we have \( u = e^3 - 1 + 1 = e^3 \).

We may now rewrite the given integral

\[
\int_{1}^{e^3-1} \frac{1}{t+1} \, dt = \int_{2}^{e^3} \frac{1}{u} \, du
\]

\[
= \ln (|u|) \bigg|_{2}^{e^3}
\]

\[
= \ln (e^3) - \ln(2) \quad \text{both endpoints are positive}
\]

\[
= 3 - \ln(2).
\]

(b) Set \( u = \frac{\pi}{x} + 1 = \pi x^{-1} + 1 \). Then \( du = -\pi x^{-2} \, dx \implies x^{-2} \, dx = \frac{du}{-\pi} \).

Rewriting the integral, we obtain

\[
\int x^{-2} e^{\pi/x+1} \, dx = \int e^{\pi/x+1} x^{-2} \, dx
\]

\[
= \int e^u \cdot \frac{du}{-\pi}
\]

\[
= -\frac{1}{\pi} \int e^u \, du
\]

\[
= -\frac{1}{\pi} [e^u + C]
\]

\[
= -\frac{1}{\pi} e^{\pi/x+1} + C,
\]

where technically we renamed \( -\frac{1}{\pi} C \) as \( C \) again.

(c) Set \( u = \ln(1-x) \) so that

\[
du = \frac{1}{1-x} \cdot (-1) \, dx \implies \frac{1}{1-x} \, dx = -du.
\]

When \( x = 0 \) we have \( u = \ln(1-0) = \ln(1) = 0 \) still, whereas when \( x = 1 - e^{-2} \), we have

\[
u = \ln(1-(1-e^{-2})) = \ln(e^{-2}) = -2.
\]
We may rewrite the integral

\[
\int_0^{1-e^{-2}} \frac{\ln(1-x)}{1-x} \, dx = \int_0^{1-e^{-2}} \ln(1-x) \cdot \frac{1}{1-x} \, dx
\]

\[
= \int_0^{-2} u \cdot (-du)
\]

\[
= - \int_0^{-2} u \, du
\]

\[
= \int_{-2}^0 u \, du
\]

\[
= \left. \frac{u^2}{2} \right|_{-2}^0
\]

\[
= \frac{0^2}{2} - \frac{(-2)^2}{2}
\]

\[
= 0 - \frac{4}{2}
\]

\[
= -2.
\]