Consider

\[ y = f(t) = b^t \]

b is called the base

Now consider exponential functions that have the base \( e = 2.71828 \ldots \)

that is,

\[ y = f(t) = e^t \]

Magically

\[ \frac{dy}{dt} = e^t \quad \text{and} \quad \int e^t \, dt = e^t + c \]

Now consider

\[ y = Ae^{rt} = Ae^{w} \quad \text{where} \quad w = rt \quad \Rightarrow \]

\[ \frac{dy}{dt} = \frac{dy}{dw} \frac{dw}{dt} = Ae^w \frac{dw}{dt} = rAe^{rt} \]

and

\[ \int rAe^{rt} \, dt = rA \int e^{rt} \, dt \]

\[ = Ae^{rt} + c \]

Note the following,

\[ \% \Delta y \quad \text{wrt} \quad t = \frac{\frac{dy}{dt}}{y} = \frac{rAe^{rt}}{Ae^{rt}} = r \]

so if \( y = Ae^{rt} \) where \( t \) is time

y is growing (or declining) at the constant % rate \( r \).
For example,

\[ s(t) = s(0)e^{0.07t} \]

where

\( s(t) \) is number of squirrels on campus at time \( t \).

In which case, the squirrel population grows at 7% a year.

Or

If the interest rate is \( r \) and \( V(0) \) dollars are invested when \( t = 0 \) then

\[ V(t) = V(0)e^{rt} \]

where \( V(t) \) is the value of the investment at time \( t \).

What then is the present value, \( PV \), of \( V(t) \) where \( t=0 \) is the present?

If \( V(t) = V(0)e^{rt} \)

then

\[ PV = V(0) = V(t)e^{-rt} \]

PV is the present value of \( V(t) \); i.e. it is what \( V(t) \) is worth to you today.

With all this in mind, consider the problem of deciding when to drink (or sell) a bottle of good wine

where

\[ V(t) = Ke^{1.5} \quad \Rightarrow \text{growth rate in the value of the wine} = (0.5t^{-5}). \]

\( V(t) \) is the value of the wine at time \( t \).

\( K \) is the purchase price.

Assume no storage costs and that the market rate of interest is \( r \).

What is the objective?

To max the value of the wine? NO

To max the PV of the wine

\[ PV(t) = V(t)e^{-rt} \]

where \( PV(t) \) is the PV of the wine if it is sold (drank) at time \( t \).

Therefore

\[ PV(t) = Ke^{1.5}e^{-rt} = Ke^{(t^5-rt)}. \]
We want to find that \( t, t^* \) that max \( PV(t) \)

\[
1. \quad \frac{dPV(t)}{dt} = \frac{dKe^{t^* - r}}{dt} = Ke^{(t^* - r)}(.5t^{-5} - r) = PV(t)(.5t^{-5} - r) \quad \text{set} = 0
\]

Since \( PV(t) \neq 0 \), \( 1 \) is only zero if \( .5t^{-5} - r = 0 \).

Solve this for \( t^* \) to obtain \( t^* = \frac{1}{4r^2} \). (Note: \( t^* \) does not depend on \( K \)).


or graphically,

When would you sell (drink) the wine if \( V(t) = Ke^{gt} \)?

Now let's consider streams of benefits (or costs).
Assume a stream of benefits \( B(t) \) \( t = 0 \) ..... 
Assume a stream of costs \( C(t) \)

What is the PV of this stream if the interest rate is \( r \)? It is

\[
PV = \int_0^{\infty} [B(t) - C(t)]e^{-rt} \, dt
\]

Example: what if you won the Colorado lottery to the tune of $1,000,000 (payable at $50,000 yr for 20 years). What are the winnings worth to you if your interest rate is \( r = .10 \)?
Do we have truth in advertising?

What if they paid $50,000/year forever (they won't)?

A constant payment that goes on forever is called a "Perpetual Flow."

What is the PV of a Perpetual Flow of constant benefits, B?

\[
\begin{align*}
PV &= \int_0^\infty Be^{-rt}dt = \lim_{s \to \infty} \int_0^s Be^{-rt}dt \\
&= B \lim_{s \to \infty} \int_0^s e^{-rt}dt \\
&= B \lim_{s \to \infty} -\frac{1}{r} (1-e^{-rs}) = B \frac{1}{r}
\end{align*}
\]

So if B = 50,000 and r = .10

PV of a perpetual flow of $50,000 is $500,000.

With a discount rate of 10% forever is not worth much more than twenty years.

What is the PV of $50,000 starting 21 years from now and lasting forever?
Now return to the wine problem where

\[ V(t) = 15e^{t^5} \]

but now assume it costs $1/yr to store the wine.

When should the wine be sold (or drank)?

There is now a stream of storage costs so, if the wine is sold at time \( t \), the PV of the storage costs are

\[ \int_0^t e^{-rt} dt = \frac{1}{r} (1 - e^{-rt}). \]

Therefore the PV of the wine if it is sold at time \( t \) is

\[ \text{max wrt } t \]

\[ PV(t) = 15e^{t^5}e^{-rt} - \frac{1}{r} (1 - e^{-rt}) \]

\[ \frac{dPV(t)}{dt} = (.5t^{-5}) 15e^{t^5}e^{-rt} - r 15e^{t^5}e^{-rt} - e^{-rt} \]

\[ = \left[ .5t^{-5} 15e^{t^5} - r 15e^{t^5} - 1 \right] e^{-rt} \text{ set } = 0 \]

but \( e^{-rt} \neq 0 \), so

\[ .5t^{-5} 15e^{t^5} - r 15e^{t^5} - 1 = 0. \]

Give \( r \) a specific value and use Mathematica to solve (use Find Root, Solve can't do it).

For example, if \( r = .01 \)

\[ .5t^{-5} 15e^{t^5} - .15e^{t^5} - 1 = 0 \]

\[ 7.5t^{-5} e^{t^5} - .15e^{t^5} - 1 = 0 \]

\[ t = .861454 \]

Storage costs are high relative to the rate of appreciation in selling price, so don't hold it
Note that, in general,
\[
\left( \frac{.5t^{-5} 15e^{t^5}}{V'(t)} - \frac{r 15e^{t^5}}{rV(t)} - 1 \right) = 0
\]

is
\[
\frac{V'(t)}{rV(t)} - s = 0
\]

increase in selling price if wait one more year
lost interest because the sale was postponed one year (could have sold and put money in bank)
storage costs

sell when ⇒
\[
\frac{V'(t)}{rV(t)} + s = 0
\]
benefits from waiting costs of waiting

⇒
\[
V'(t) - s = rV(t)
\]
\[
r = \frac{V'(t)}{V(t)} - \frac{s}{V(t)}
\]
% grow rate in the value of the wine (accounting for storage costs)

Graphically,
In our particular case

\[
\frac{V'(t)}{V(t)} - \frac{s}{V(t)} = \frac{.5t^{-5} 15e^t}{15e^t} - \frac{1}{15e^t} = .5t^{-5} - \left(15e^t\right)^{-1}
\]

- % growth rate in wine not accounting for storage costs
- % growth rate in wine accounting for storage costs