Reviewing
\[ \int f(x) \, dx = F(x) + C = G(x) \]

is called an indefinite integral. \[ \int f(x) \, dx \] is called an indefinite integral, rather than a definite integral, because \( F(x) + C \) is a function rather than a specific (definite) number.

Now consider this function
\[ G(x) = F(x) + C \]
evaluated at two specific values of \( x \), \( a \) and \( b \)
where \( b > a \)
i.e.
if \( x = b \) \( F(x) + C = F(b) + C \)
if \( x = a \) \( F(x) + C = F(a) + C \).

Now consider the difference between \( [F(b) + C] \) and \( [F(a) + C] \)
\[ [F(b) + C] - [F(a) + C] = F(b) - F(a) = F(x) \left|_{a}^{b} \right. \]
Note that \( C \) cancels out.

\( F(b) - F(a) \) is called the definite integral of \( f(x) \) from \( a \) to \( b \).
(Note that \( F(b) - F(a) = G(b) - G(a) \).)

We now need a notation for \( F(b) - F(a) \) in terms of \( f(x) \), \( a \), and \( b \).

The standard notation is
\[ \int_{a}^{b} f(x) \, dx = F(x) \left|_{a}^{b} \right. = F(b) - F(a) \]
i.e. \( \int \) is modified by adding \( a \) and \( b \) to get \( \int_{a}^{b} \).

Note that \( b \) is referred to as the upper limit of integration and \( a \) is referred to as the lower limit of integration.
How should we interpret

\[ \int_{a}^{b} f(x) \, dx = F(x) \big|_{a}^{b} = G(x) \big|_{a}^{b} \]

It is the indefinite integral \( G(x) \) evaluated at \( x = b \) minus \( G(x) \) evaluated at \( x = a \).

What this means can be clarified with a few examples.

Consider our previous marginal cost example where

\[ MC(x) = 3x^{1.5} \text{ and } C(0) = 14 \]

Determine and interpret

\[ \int_{a}^{b} MC(x) \, dx \]

Before starting, remember that

\[ \int MC(x) \, dx = \int 3x^{1.5} = 1.2x^{2.5} + A \]

which combined with \( C(0) = 14 \)

\[ \Rightarrow C(x) = 1.2x^{2.5} + 14 \]

therefore

\[ \int_{a}^{b} MC(x) \, dx = C(x) \big|_{a}^{b} = C(b) - C(a) \]

\[ \equiv \Delta \text{ in total min costs when output is } \uparrow \text{ from } x = a \text{ to } x = b. \]

In this specific case

\[ \int_{a}^{b} MC(x) \, dx = \int_{a}^{b} 3x^{1.5} \, dx = (1.2x^{2.5}) \big|_{a}^{b} = 1.2b^{2.5} - 1.2a^{2.5} = 1.2(b^{2.5} - a^{2.5}) \]

\[ \equiv \text{ how much total min cost will } \uparrow \text{ if output is } \uparrow \text{ from } a \text{ to } b. \]
If, for example, $a = 1$ and $b = 5$

\[
\int_{1}^{5} MC(x) \, dx = \int_{1}^{5} 3x^{1.5} \, dx = 1.2[(5)^{2.5} - 1] = 1.2[55.9 - 1] = 65.88
\]

= how much total min costs will ↓ if output is ↓ from 1 to 5.

Now consider our earlier production example

\[
MP_L = .4L^{-6} \bar{K}^6
\]

where \( MP_L = \frac{\partial f(\bar{K}, L)}{\partial L} \).

Earlier we determined that if \( MP_L = .4L^{-6} \bar{K}^6 \)

\[
\int MP_L \, dL = \int \frac{\partial f(\bar{K}, L)}{\partial L} \, dL = \bar{K}^6 L^4 + B(\bar{K})
\]

So,

\[
f(\bar{K}, L) = \bar{K}^6 L^4 + B(\bar{K})
\]

therefore

\[
\int_{a}^{b} MP_L \, dL = f(\bar{K}, L)\Big|_{a}^{b} = f(\bar{K}, b) - f(\bar{K}, a)
\]

= how much max output ↓ when L is ↓ from $L = a$ to $L = b$ with $K = \bar{K}$.

In this specific case

\[
\int_{a}^{b} MP_L \, dL = \int_{a}^{b} .4L^{-6} \bar{K}^6 \, dL = \bar{K}^6 L^4\Big|_{a}^{b} = \bar{K}^6 b^4 - \bar{K}^6 a^4 = \bar{K}^6 (b^4 - a^4)
\]

= how much max output ↓ when L is ↓ from $L = a$ to $L = b$ with $K = \bar{K}$. 
For example, if $K = 10$

\[
\int_{3}^{7} MP_L \, dL = \int_{3}^{7} 4L^{-6} \, 10^{-6} = 10^{-6}(7^4 - 3^4) = 3.98(2.18 - 1.55) = 3.98(.626) = 2.49
\]

i.e. output ↑ by 2.49 when L is ↑ from 3 to 7 with $K = 10$.

So, in general, how should we interpret

\[
\int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = G(x) \bigg|_{a}^{b}.
\]

It's the primitive function, $G(x)$, evaluated at $x = b$ minus the primitive evaluated at $x = a$; i.e. how much the value of the primitive function ↑ (or ↓) in value when $x$ ↑ from $a$ to $b$.

Graphically, if

![Graphical representation of integration](image)

Note that

\[
\int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = G(x) \bigg|_{a}^{b} = G(b) - G(a)
\]

can also be interpreted as the area under the $f(x)$ function between $a$ and $b$ and above the horizontal axis.
i.e.

\[ \int_{a}^{b} f(x) \, dx \]
equals the shaded area.

Therefore, the vertical distance \( G(b) - G(a) \) equals this shaded area.

How do we know that

\[ \int_{a}^{b} f(x) \, dx \]
can be interpreted as an areas under \( f(x) \) between \( a \) & \( b \) and above the horizontal axis?

I don't want to prove it. Rather, I'll just provide a loose, but hopefully intuitive, explanation.

Consider the general problem of finding the area under some function, \( f(x) \), between \( x = a \) and \( x = b \).
The area under this curve is approximated by

\[ \sum_{i=1}^{N} f(x_i) \Delta x \]

where

\[ N \Delta x = b - a \]

\[ x_1 = a \]

\[ x_{N+1} = b \]

So, for example if \( N = 4 \)

(b - a) is divided into 4 segments of equal length

and

\[ \sum_{i=1}^{4} f(x_i) \Delta x \]

equals the shaded area

This is an approximation to the area under \( f(x) \) from \( a \) to \( b \).

Not a great approximation, but an approximation.

Note however the approximation will improve as \( N \uparrow (\Delta x \downarrow) \) and in the limit will equal the area under the curve.
i.e.
\[ \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x = \text{area under the curve} \]

where
\[ x_1 = a \]
\[ x_{N+1} = b \]

and
\[ N \Delta x = b - a \]

The final step in the proof is to demonstrate
\[ \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x = \int_{a}^{b} f(x) \, dx. \]

I won't do this, but note it makes sense if one notes that \( dx \) denotes an infinitely small change in \( x \) and interprets \( \int_{a}^{b} f(x) \, dx \) as the sum of an infinite number of terms.

So, in our marginal cost example

**Note that graph on left has wrong shape**

The shaded area = \( C(b) - C(a) = \int_{a}^{b} MC(x) \, dx \),
and in our marginal product example

![Graph showing marginal product](image)

the shaded area = $f(\bar{K}, b) - f(\bar{K}, a)$.

Make sure you are familiar with the properties of definite integrals (Chiang, pages 451-453).

In conclusion, consider one special case of definite integrals.

When either

$$\int_{a}^{b} f(x) \, dx \quad \text{or} \quad \int_{-\infty}^{b} f(x) \, dx \quad \text{or} \quad \int_{a}^{\infty} f(x) \, dx$$

i.e. one of the limits of integration is infinite.

In this case

$$F(\infty) - F(a), F(b) - F(-\infty), \text{ and } F(\infty) - F(-\infty)$$

cannot be evaluated because $\infty$ is not a number.

However, one can consider

$$\lim_{x \to \infty} F(x)$$

The integrals

$$\int_{a}^{\infty} f(x) \, dx \quad \int_{-\infty}^{b} f(x) \, dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \, dx$$

are deemed improper integrals.
Define the improper integral as the limit of a proper integral, i.e.

\[
\int_{a}^{b} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \quad \text{if this limit exists}
\]

and

\[
\int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \quad \text{if this limit exists}
\]

and

\[
\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty} \lim_{b \to +\infty} \int_{a}^{b} f(x) \, dx \quad \text{if both these limits exist.}
\]

These limits will not always exist.

- If the limit exists the improper integral is said to be convergent.
- If the limit does not exist the improper integral is said to be divergent and the improper integral does not have a definite value.

For example, consider the improper integral

\[
\int_{0}^{\infty} Re^{-nt} \, dt = \lim_{b \to \infty} \int_{0}^{b} Re^{-nt} \, dt
\]

Note that \(\frac{de^t}{dt} = e^t\) and \(\frac{d(e^r)}{dt} = re^r\)

\[
= \lim_{b \to \infty} \left( - \frac{R}{r} e^{-nt} \bigg|_{0}^{b} \right)
\]

\[
= \lim_{b \to \infty} \left( - \frac{R}{r} \left[ e^{-rb} - e^{0} \right] \right) = \lim_{b \to \infty} \frac{R}{r} \left[ 1 - e^{-rb} \right]
\]

\[
= \frac{R}{r} \quad \text{because} \quad \lim_{b \to \infty} e^{-rb} = 0
\]

So the area under \(Re^{-nt}\) from 0 to \(\infty\) is finite and equals \(\frac{R}{r}\).
Alternatively, consider the improper integral

\[ \int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} \, dx \]

\[ = \lim_{b \to \infty} \ln x \bigg|_{1}^{b} = \lim_{b \to \infty} [\ln b - \ln 1] \]

\[ = \lim_{b \to \infty} \ln b = \infty. \]

So this improper integral is divergent and the area under \( \frac{1}{x} \) between 1 and \( \infty \) is infinite.