Introduction to integration with some simple economic examples - Lecture 1

Topics covered:
• derivatives & antiderivatives (integrals)
• the notation of integration
• indeterminacy of integration
• indefinite vs. definite integration
• some simple economic examples
• initial conditions.

Consider two functions of $G(x) = F(x) + C$ and $f(x)$, where $x$ is a scalar and where
\[
\frac{dG(x)}{dx} = \frac{dF(x)}{dx} = f(x)
\]
i.e. $f(x) = F'(x) = G'(x)$.

Refer to $G(x)$ as the primitive function.
In differential calculus we start with the primitive, $G(x)$, and differentiate to derive $f(x)$.

One might view $f(x)$ as the child of the single parent $G(x)$; that is, there is sufficient information in $G(x)$ to completely determine $f(x)$.

Now I want you to think about the opposite procedure where you know $f(x)$ and you are trying to determine $G(x)$;

i.e. trying to identify the single parent, $G(x)$, on the basis of what the kid, $f(x)$, looks like.

The procedure to do this might be deemed antidifferentiation;
e.g. assume $f(x) = 2x$.

Note that $\frac{d(x^2)}{dx} = 2x$, so one might conclude that $x^2$ is the antiderivative of $2x$. 
However, further note that \( \frac{d(x^2 + 4)}{dx} = 2x \), so \( x^2 + 4 \) is also an antiderivative of \( 2x \);

that is, in general, \( \frac{d(x^2 + C)}{dx} = 2x \) so \( x^2 + C \) is the antiderivative of \( 2x \).

i.e. if \( f(x) = 2x \)
then
\[ G(x) = F(x) + C \]
where
\( F(x) = x^2 \) and \( C \) is an unknown constant.

Another name for antidifferentiation is integration.

If one starts with \( G(x) \) one differentiates with reference to \( x \) to get \( f(x) \).

Alternatively, if one starts with \( f(x) \) one can integrate with reference to \( x \) to get \( F(x) + C \).

The function that one is integrating is referred to as the integrand; that is, \( f(x) \) is the integrand.
\( F(x) \) is the integral, or antiderivative of \( f(x) \).
\( C \) is called the constant of integration.

Before proceeding, note that \( G(x) \) contains sufficient information to determine \( f(x) \);

that is, a function, \( G(x) \) completely determines its derivative \( \frac{dG(x)}{dx} = f(x) \).

But that \( f(x) \) does not contain sufficient information to determine \( G(x) \); that is, by integration of \( f(x) \) one can completely determine \( F(x) \) but not \( G(x) \) because one cannot determine \( C \) from \( f(x) \).

In summary, one can determine the kid if one knows the single parent but one cannot identify the single parent just on the basis of what the kid looks like.
To identify $G(x)$,

one needs to know $f(x)$ and the specific value of $G(x)$ at at least one value of $x$;

\[ \text{e.g. in my example} \]

if $f(x) = 2x$

and, in addition, one knows $G(0) = 7.2$

then $G(x) = x^2 + 7.2$

the assumption that $G(0) = 7.2$ is called an initial condition.

Any time we know the value of $G(x)$ at some specific $x$, we have an initial condition on $G(x)$

which we can use, along with $F(x)$, to determine the value of $C$.

Now let’s introduce some notation.

We use $\frac{dG(x)}{dx}$ to denote the derivative of $G(x)$ with reference to $x$.

What notation should we use to denote the integral of $f(x)$ with reference to $x$?

The convention is $\int f(x) \, d(x)$

where $\int$ is called the integral sign.

But since we have defined $G(x)$ as the primitive function for which $\frac{dG(x)}{dx} = f(x)$

$\int f(x) \, d(x) = F(x) + C = G(x)$

where the indeterminate $C$ indicates that $f(x)$ could have come from many different parents.

The integral $\int f(x) \, d(x)$ is more specifically called an indefinite integral because it generates, through the process of integration, a function rather than a specific numerical value.

Now that we know what an integral is, let’s try to determine how one integrates.

Put simply, one reverses the process of differentiation.
This means one integrates by using the rules for differentiation and reversing them.

These reverse rules are adequately explained on pages 438-446 in Chiang.

Let's just look at his Rule I;

By differentiation,

$$\frac{d}{dx} \left( \frac{x^{n+1}}{n+1} \right) = x^n \quad \text{where } n \neq -1;$$

therefore,

$$\int x^n \, dx = \left( \frac{1}{n+1} \right) x^{n+1} + C.$$ 

This is called the power rule.

Review all of the rules of integration and make sure you can apply them.

A quiz with examples or problems from Chiang is likely.

Let's add to Chiang's Rule for integration the Edward Rule which is better known as the Guessing Rule.

I have $f(x)$ but only vaguely remember the rules (I don't find them as intuitive as the rules of differentiation), so

I guess at $F(x)$, then check to see whether my guess is correct by differentiating my guess with reference to $x$.

If this derivative is $f(x)$, I know I have guessed correctly.

If it's not correct, I modify my guess.

This process iterates until I have either found the correct solution or I give up.

Recently I have discovered another good rule - Ask Mathematica.
Now that we know everything there is to know about integration, let's use it to solve some economic problems.

Assume \( MC(x) = 3x^{1.5} \)

where \( MC(x) \) is the marginal cost function for producing \( x \).

Using \( MC(x) \), determine as much as you can about the cost function \( C(x) \)

\[
\int MC(x) \, dx = \int 3x^{1.5} \, dx = 3 \int x^{1.5} \, dx
\]

\[
= 3 \left[ \frac{x^{2.5}}{2.5} \right] + A = 1.2 \, x^{2.5} + A
\]

that is,

\( C(x) = 1.2 \, x^{2.5} + A \)

i.e. we have determined \( C(x) \) up to the unknown constant of integration, \( A \).

Now assume the following initial condition,

\( C(0) = 14 \)

it's the short-run therefore

\( C(0) = 1.2 \, (0)^{2.5} + A = 14 \)

\[ \rightarrow A = 14 \]

and

\( C(x) = 1.2 \, (x)^{2.5} + 14 \)
Now consider the SR production function \( y = f(K, L) \)

where we don't know \( f(K, L) \) but we do know that \( MP_L = .4L^{-6}K^{.6} \)

where \( MP_L \) is the Marginal Product of Labor.

Determine using \( MP_L \) as much as you can about the SR production function \( f(K, L) \).

Then figure out what value the constant of integration must take in this case.

\[
\int MP_L \, dL = \int (.4L^{-6}K^{.6}) \, dL = .4 \frac{L^{.4}}{4} + B(K)
\]

\[
= .4K^{.6}L^{.4} + B(K)
\]

where \( B \) is the unknown constant of integration that can depend on \( (K) \); i.e. \( B(K) \).

In this case, can we determine \( B(K) \)?

No, physics tells us

\( f(0, 0) = 0 \)

so

\( f(0, 0) = 0^6 0^4 + B(0) = 0 \) but this does not imply that \( B(K) = 0 \)

The issue is whether \( L \) is an essential input (required for production). If it is, then \( B(K) = 0 \). If not, then \( B(K) \neq 0 \) because it is possible to produce with only \( K \).