Econ 6806, Final, December 10, 1999

Answer the questions to the best of your ability in the allotted time. Make sure you have all ten questions. The exam consists of two pages.

You can choose 20 points worth of questions for which you will be given full credit without answering the question. Specify clearly the questions for which you want automatic credit by writing "full credit" next to the question number in your blue book.

Thank you for being in my class. I enjoyed the experience.

Please use the provided blue books for your answers.

1. (10 points) Consider OLS estimation of the parameters in the linear regression equation $y = a + \beta x + e$, where epsilon is a random term with an expected value of zero. Identify the OLS estimates of $a$ and $\beta$ as the solution to a Minimization problem. Assume at least two observations using actual numbers for the data.

Let $\hat{y}_i$ denote the predicted value of $y$, where $\hat{y}_i = \hat{\alpha} + \hat{\beta} x_i$, $i = 1, 2$, and $\hat{\alpha}$ and $\hat{\beta}$ are the OLS estimates of $\alpha$ and $\beta$ respectively.

OLS minimizes the sum of squared residuals, so the minimization problem is

$$ \min_{\alpha, \beta} \left[ (y_1 - \alpha - \beta x_1)^2 + (y_2 - \alpha - \beta x_2)^2 \right] $$

Find the min by taking the partial derivatives of the objective function wrt to $\alpha$ and $\beta$, set them equal to zero and solve for $\hat{\alpha}$ and $\hat{\beta}$ - the OLS estimates.

For example, if the respective values of $y$ are 10 and 4 and the respective values of $x$ are 2 and 1, the objective function to minimize is

$$ \min_{\alpha, \beta} \left[ (10 - \alpha - \beta 2)^2 + (4 - \alpha - \beta)^2 \right] $$

Find the partials, set em equal to zero, and solve for the OLS estimates.
2. (10 points) Intuitively explain, using a graphical analysis, how one can derive the production function, \( y = f(k, l) \) from the cost function \( c = c(y, r, w) \).

I can’t draw it in WordPerfect, but I wanted you to do the following. Choose and output level, e.g. \( y=1 \). Then use the cost function with specific values of \( r \) and \( w \), determine min cost, and draw the isocost line for this cost and input prices. Given the definition of the cost function, we know the isoquant for \( y=1 \) cannot lie to the left of this isocost line and must touch it at at least one point. This tell us something about the position of the isoquant for \( y=1 \).

Repeat for different input prices.

Doing this will trace out the isoquant for \( y=1 \).

This exercise could, in theory, be repeated for all values of \( y \). In which case, one then would know the shape and position of all of the isoquants, so one would know the production function.

3. (10 points) Define *Shephard's Lemma* in terms of consumer theory. What is derived and what does it mean?

I assume only two goods.

Define the expenditure function

\[
E = E(U, p_1, p_2)
\]

as the minimum expenditures needed to achieve utility level \( U \) given the prices of the two goods.

Define the Hicksian demand function for good \( i \) as

\[
x_i = x_i(U, p_1, p_2)
\]

as the amount of good \( i \) the individual will purchase to minimize the cost of achieving utility level \( U \) given the prices of the two goods.

In words, Shepard’s Lemma tells us that the Hicksian demand function for good \( i \) is the partial derivative of the expenditure function \( \frac{\partial E(U, p_1, p_2)}{\partial p_i} \) = \( x_i(U, p_1, p_2) \) \( i = 1,2 \).
4. (10 points) Demonstrate, using graphs or whatever, that the production function \( X = L^{0.5}K^{0.5} \), is strictly quasi-concave in terms of \( L \) and \( K \). Hint: think about the shape of the isoquants for this production function and whether this production function is increasing in its arguments. As part of your answer define quasi-concavity of the production function in terms of \( L \) and \( K \).

The production function is quasi-concave is all of it upper-level sets are convex sets.

Since the partial derivatives of this production function are both positive, this production function is strictly increasing in \( L \) and \( K \), so the upper level sets lie to the right of the isoquants.

I would finish by demonstrating that the isoquants have a negative slope that becomes less negative as \( L \) increases (\( L \) on the horizontal axis), which shows that the upper-level sets are convex.

I would derive the slope by taking the total differential of the production function, set it equal to zero (because along the isoquant output does not change) and solve it for \( dK/dL \).

5. (10 points) What is a continuous random variable? Start your answer with the statement, "\( x \) is a continuous random variable if ...". Define, both in words and functional notation, the joint density function for the random variables \( x \) and \( y \). Also define in both words and functional notation the marginal and conditional distributions for \( x \).

The variable \( x \) is a continuous random variable if

\[
\Pr \{a \leq x \leq b\} = \int_a^b f(x) \, dx
\]

where \( f(x) \geq 0 \forall x \) and \( \int_{-\infty}^{+\infty} f(x) \, dx = 1 \).

\( x \) and \( y \) are jointly distributed random variables if

\[
\Pr \{a \leq x \leq b, c \leq y \leq d\} = \int_a^b \int_c^d f(x, y) \, dy \, dx
\]

where \( f(x, y) \geq 0 \forall x, y \) and \( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \, dy \, dx = 1 \).

\( f(x, y) \) is the joint density function for \( x \) and \( y \). It tells us the joint distribution of these two random variables.
The marginal distribution of $x$ is
\[ f_x(x) \equiv \int_{-\infty}^{+\infty} f(x, y) dy \]
The marginal distribution of $x$ is, simply put, the distribution of $x$ over all values of $y$. In terms of the area under the joint density, the height of the marginal density function of $x$ evaluated at $x = a$ is the area under the joint density when $x=a$ for all values of $y$.

The conditional distribution of $x$, conditional on $y=c$, is
\[ f(x|y=c) = f(x,c) / f_y(c) \]
The conditional distribution of $x$ given a specific value of $y$ is, simply put, the distribution of $x$ given a specific value for $y$. It first blush, one might assume $f(x|y=c) = f(x,c)$ but this would be incorrect because $\int_{-\infty}^{+\infty} f(x,c) dx \neq 1$. Dividing $f(x,c)$ by the marginal density of $y$ evaluated at $c$ “adjusts” it so the area is one.

6. (10 points) Consider the multivariate density function $f(x,y) = 1/8(6 - x - y)$ if $0 < x < 2$ and $2 < y < 4$ and zero otherwise. Find the marginal density of $y$, and the conditional density of $y$, conditional on $x$.

\[ f_y(y) = \int_{-\infty}^{+\infty} f(x,y) dx = \frac{1}{8} \int_{-\infty}^{+\infty} (6 - x - y) dx \]
\[ = \frac{1}{8} [(6x - (1/2)x^2 - xy)]_0^4 = \frac{5}{4} - \frac{y}{4} \]

\[ f(y|x = c) = f(c,y) / f_x(c) \]

First determine
\[ f_x(x) = \int_{-\infty}^{+\infty} f(x,y) dy = \frac{1}{8} \int_{-\infty}^{4} (6 - x - y) dy \]
\[ = (6y - xy - (1/2)y^2)]_2^4 = 6 - 2x \]

So,
7. (10 points) Assume the cost function for a competitive firm is
\[ c(x, w, r) = x^{4} g(w, r) \]
where
\[ g(w, r) > 0 \]
Derive the profit maximizing level of output, \( x^* \)

Price is fixed and marginal cost is continuously decreasing, so the interior solution minimizes profits. Profits are continuously increasing in output, supply is infinity.

8. (10 points) Make-up a density function, prove that it is a density function, and derive its CDF.

Very simply,
\[ f(x) = 1 \text{ if } 0 \leq x \leq 1 \]
\[ f(x) = 0 \text{ otherwise} \]

It is obvious that the function is always nonnegative.

Show that the area under it is one.

The CDF is
\[ F(x) = \int_{-\infty}^{x} f(t) dt \]
\[ = 0 \text{ if } x < 0 \]
\[ = \int_{0}^{x} 1 dt = x \text{ if } 0 \leq x \leq 1 \]
\[ = 1 \text{ if } x \geq 1 \]
9. (10 points) Derive the indirect utility function, \( U = V(m, p_1, p_2) \), assuming the following hicksian demand functions: 

\[
\begin{align*}
    x_1^h &= \alpha U p_1^{(\alpha-1)} p_2^\beta, \\
    x_1^h &= \beta U p_1^\alpha p_2^{\beta-1}
\end{align*}
\]

As you all knew

\[
E(U, p_1, p_2) = p_1 x_1(U, p_1, p_2) + p_2 x_2(U, p_1, p_2)
\]

10. (10 points) Consider a random variable \( x \) with density \( f(x) \). Will, in general, \( E[x^2] = (E[x])^2 \)? How, in general, would one determine the expected value of \( x \), \( E[x] \), and the expected value of \( x^2 \), \( E[x^2] \). Use some specific density function and determine \( E[x] \), \( E[x^2] \) and \( (E[x])^2 \).

No.

\[
E[g(x)] = \int_{-\infty}^{+\infty} g(x) f(x) dx
\]

So

\[
E[x] = \int_{-\infty}^{+\infty} x f(x) dx \quad \text{and}
\]

\[
E[x^2] = \int_{-\infty}^{+\infty} x^2 f(x) dx
\]

So

\[
(E[x])^2 = \left[ \int_{-\infty}^{+\infty} x f(x) dx \right]^2
\]

For my density function in question 8

\[
E[x] = \int_0^1 x^2 dx = \frac{x^3}{3}\big|_0^1 = 1/3 \quad \text{So} \quad (E[x])^2 = 1/9
\]

But
\[ E[x^2] = \int_0^1 x^3 dx = \left. \frac{x^4}{4} \right|_0^1 = \frac{1}{4} \]

Big difference.