1. What is my maximum willingness to pay for another bottle of scotch in terms of pounds of Swiss chocolate if my utility function is \( u = u(s, c) = c^2s \)?

2. Answer: Need to find \( \left. \frac{dc}{ds} \right|_{du=0} = -\frac{dc}{uc} = -\frac{c^2}{2cs} = -\frac{cc}{2cs} = -\frac{c}{2s} \). That is, I will give up up to \( c/2s \) pounds of chocolate to get another bottle of scotch. To get a numerical answer, one needs to know how much scotch and chocolate I am currently consuming. For example, if I am currently consuming 4 pounds of chocolate and 3 bottles of scotch, I will pay a maximum of \( 2/3 \) of a pound of chocolate to get another bottle of scotch. Continuing with stuff unasked, willingness to pay, \( \text{wtp} \), for scotch in terms of chocolate is the function \( \text{wtp}(c, s) = c/2s \). So, for example, if one asked how much my \( \text{wtp} \) would change if my chocolate consumption marginally increased, one would take the derivative of \( \text{wtp}(c, s) \) wrt to \( c \) and get \( 1/2s \).

3. Marvin consumes just two goods; Tofu and Soy Milk. Let \( T \) denote the pounds of Tofu that he consumes and let \( S \) denote the gallons of Soy Milk that he drinks. Assume that his preferences for these two goods can be represented by the utility function \( U = aT^\alpha S^\beta \) where \( \alpha, \beta > 0 \).

Determine, in percentage terms, Marvin’s maximum willingness to pay for Tofu in terms of Soy milk. Show, and explain in words, all of your work. Make sure to explain your answer in words.

Are Tofu and Soy Milk perfect substitutes? Could they be if one restricts the parameter values \((\alpha, \alpha, \beta)\) in certain ways? Explain.

Answer: \((\alpha/\beta)\%\), that is, Marvin will be willing to give up \((\alpha/\beta)\%\) of his current consumption of Soy Milk to obtain a 1% increase in his Tofu consumption. No. They cannot be perfect substitutes. Even though the \( MRS_{TS} \) is constant in % terms, it is not constant in terms of the quantities of Tofu and Soy Milk. The derivation proceeds as follows: First determine

\[
\left. \frac{dS}{dT} \right|_{dU=0}
\]

This is how much soy milk Marvin will give up to obtain one more unit of tofu.

\[
\left. \frac{dS}{dT} \right|_{dU=0} = -\frac{U_T}{U_S} = -\frac{aaT^\alpha S^\beta T^{-1}}{\beta aT^\alpha S^\beta S^{-1}} = -\frac{aS}{\beta T}
\]

That is to obtain one more unit of tofu Marvin will give up \( \frac{aS}{\beta T} \) units of soy milk. Now convert this into a percentage change

\[
\left. \frac{\% \triangle S}{\% \triangle T} \right|_{dU=0} = \left. \frac{dS}{dT} \right|_{dU=0} \frac{T}{S} = -\frac{aS}{\beta T} \frac{T}{S} = -\frac{a}{\beta}
\]
That is, Marvin will be willing to give up $\frac{\alpha}{\beta}$% of his soy milk to get 1% more tofu.

4. Read this question carefully. Assume a world of just two goods, $x_1$ and $x_2$ where the consumer attempts to choose that combination of $x_1$ and $x_2$ that maximizes her utility, $u(x_1, x_2)$, subject to the budget constraint, $y = p_1 x_1 + p_2 x_2$

where $y$ is income and $p_i$ is the price of good $i$.

Note that the budget equation $y = f(x_1, x_2) = p_1 x_1 + p_2 x_2$ describes all those combinations of $x_1$ and $x_2$ that the individual can just afford.

Using the concept of a total differential, determine the rate at which the market allows the consumer to substitute $x_1$ for $x_2$. Start from the beginning and explain, in words, each mathematical step.

answer: Remember that $p_1$ and $p_2$ are exogenous. Along the budget line $dy = 0$, so

$$0 = dy = p_1 dx_1 + p_2 dx_2$$

Rearranging, one gets

$$\frac{dx_2}{dx_1} \bigg|_{dy=0} = -\frac{p_1}{p_2}$$

1. Assume that there are only three things that you can do in this world: ski, sleep and attend math-econ lectures. Let $x$ denote hours of skiing, $z$ hours of sleeping and $m$ hours of attending math econ lectures. Assume that all three of these activities are "goods," rather than "bads," and that Fred's preferences can be described by some utility function

$$u = u(x, z, m)$$

Derive that rate at which Fred is willing to substitute math-econ lectures for sleep. Start from the beginning and explain, in words, each of your steps.

Now specifically assume that

$$u = u(x, z, m) = ax^a z^b m^c + bz$$

How many hours of math-econ lectures will Fred give up to get to sleep one more hour?

Specifically, how many hours of lectures will he give up to get one more hour of sleep if $a = 4$, $b = 6$, $\alpha = 2$, $\beta = .5$, $\gamma = 1$ and he is currently consuming 7 hours of skiing, 25 hours of sleep and 10 hours of lectures. Express your answer in minutes?

What if $a = 100$, $b = 25$, $\alpha = .5$, $\beta = .7$ and $\gamma = 1$ and Fred is currently consuming 10 hours of skiing, 10 hours of lectures, and 25 hours of sleep.

answer: But not from the beginning, we are looking for

$$\frac{dm}{dz} \bigg|_{dx=0, du=0}$$

which will tell us how much math econ Fred will give up to get one more hour of sleep.
\[
\begin{align*}
\frac{dm}{dz} \bigg|_{dx=0 \ dw=0} &= -\frac{\frac{\partial u}{\partial z}}{\frac{\partial u}{\partial m}} \\
&= -\left[ \frac{\beta ax^a z^{b-1} m^\gamma + b}{\gamma ax^a z^{b} m^\gamma-1} \right] \\
&= -\frac{\beta ax^a z^{b-1} m^\gamma}{\gamma ax^a z^{b} m^\gamma-1} - \frac{b}{\gamma ax^a z^{b} m^\gamma-1} \\
&= -\frac{\beta z^{-1}}{\gamma m^{-1}} - \frac{b}{\gamma ax^a z^{b} m^\gamma-1} \\
&= -\frac{\beta m}{\gamma z} - \frac{b}{\gamma ax^a z^{b} m^\gamma-1}
\end{align*}
\]

Plug in the first set of parameter values to get
\[
\frac{dm}{dz} \bigg|_{dx=0 \ dw=0} = -\frac{5m}{z} - \frac{6}{4x^2 z^5}
\]

Plug in the current amounts of sleep and lectures
\[
\frac{dm}{dz} \bigg|_{dx=0 \ dw=0} = -\frac{5 \cdot 10}{25} - \frac{6}{4(7)^2 25^5}
\]
\[
= -2 - \frac{6}{4(49)(5)}
\]
\[
= -0.20612 \text{ hrs}
\]
\[
= -12.367 \text{ min}
\]

He is willing to give up 12 + minutes of lectures to get one more hour of sleep. What if \(a = 100, b = 25, \alpha = 5, \beta = 7\) and \(\gamma = 1\) and Fred is currently consuming 10 hours of skiing, 10 hours of lectures, and 25 hours of sleep.

\[
\frac{dm}{dz} \bigg|_{dx=0 \ dw=0} = -\frac{\beta m}{\gamma z^2} - \frac{b}{\gamma ax^a z^{b} m^\gamma-1}
\]
\[
= -\frac{7m}{2} - \frac{25}{100(x)^5(z)^7}
\]
\[
= -\frac{7(10)}{25} - \frac{25}{100(10)^3(25)^7}
\]
\[
= -0.28831 \text{ hrs}
\]
\[
= -17.299 \text{ min}
\]

He will give up 17 minutes of lectures for another hour of sleep.

2. Assume the production function \(x = f(k, l)\). Derive, using the concept of a total differential,

\[
\frac{dl}{dk} \bigg|_{dx=0} =
\]

Specifically, what is it if

\[x = f(k, l) = a_0 + a_1 l^{\beta_1} + a_2 k^{\beta_2}\]

Now assume \(a_0 = 6, a_1 = 6, a_2 = 8, \beta_1 = 2\) and \(\beta_2 = 3\). In this specific case,
how much can the firm decrease its use of labor if it increases its use of capital by one unit from an initial level of 4?

**answer:**

\[ dx = f_k dk + f_l dl \]

Given that \( dx = 0 \), and rearranging, one obtains

\[ \frac{dl}{dk} \bigg|_{dx=0} = -\frac{f_k}{f_l} \]

where

\[ f_k = \beta_2 \alpha_2 k^{\beta_2 - 1} \]
\[ f_l = \beta_1 \alpha_1 l^{\beta_1 - 1} \]

Substituting these in

\[ \frac{dl}{dk} \bigg|_{dx=0} = -\frac{f_k}{f_l} = -\frac{\beta_2 \alpha_2 k^{\beta_2 - 1}}{\beta_1 \alpha_1 l^{\beta_1 - 1}} \]

Now assume \( \alpha_0 = 6, \alpha_1 = 6, \alpha_2 = 8, \beta_1 = 2 \) and \( \beta_2 = 3 \). Plugging these in

\[ \frac{dl}{dk} \bigg|_{dx=0} = -\frac{\beta_2 \alpha_2 k^{\beta_2 - 1}}{\beta_1 \alpha_1 l^{\beta_1 - 1}} = \frac{3(8)k^2}{2(6)l} = -\frac{2k^2}{l} \]

In this specific case, how much can the firm decrease its use of labor if it increases its use of capital by one unit from an initial level of 4? Plug in \( k = 4 \) to obtain

\[ \frac{dl}{dk} \bigg|_{dx=0} = -\frac{2k^2}{l} = -\frac{2(4)^2}{l} = -\frac{32}{l} \]

That is, if capital increases by 1 unit from 4 units, labor can be decreased by \( 32/l \) units. The specific numerical answer will depend on the current level of labor. If, for example, \( l = 16 \), the production manager can reduce labor by 2 units when capital in increased by 1 unit from 4 units.

**3.** Assume the production function \( x = f(k, l) \). Derive, using the concept of a total differential,

\[ \frac{dl}{dk} \bigg|_{dx=0} = \]

Specifically, what is it if

\[ x = f(k, l) = \alpha_0 + \alpha_1 l^\beta + \alpha_2 k^2 \]

Now assume \( \alpha_0 = 6, \alpha_1 = 1, \alpha_2 = 2, \beta = 2 \) and \( \beta_2 = 3 \). In this specific case, how much can the firm decrease its use of labor if it increases its use of capital by one unit from an initial level of 4?

How would you describe, in economic terminology, the relationship between labor and capital when \( \alpha_1 = 1, \alpha_2 = 2 \) and \( \beta = 1 \)?

**4.** Assume that Gertrude the recluse exists on just three goods: grapefruit (G), scotch (S), and Big Macs (M). Her preferences for these three goods can be
represented by some utility function:

\[ u = u(G, S, M) \]

Derive, using the concept of a total differential, Gertrude’s marginal rate of substitution between scotch and Big Macs, \( MRS_{SM} \)

\[
\frac{dM}{dS} \bigg|_{dG=0} = -MRS_{SM}
\]

**answer:** Find the total differential of \( u \), \( du \), set it equal to zero and solve to find

\[ MRS_{SM} = \frac{u_s}{u_m} \]

Now assume that Gertrude’s preferences can be represented by the specific utility function:

\[ u = u(G, S, M) = \alpha_0[\alpha_1 G^\rho + \alpha_2 S^\rho + \alpha_3 M^\rho]^{1/\rho} \]

where \(-1 < \rho \neq 0 \) and \( \alpha_i > 0 \ \forall i \).

Determine Gertrude’s marginal rate of substitution between scotch and Big Macs, \( MRS_{SM} \).

**answer:**

\[ u_s = (1/\rho)\alpha_0[\alpha_1 G^\rho + \alpha_2 S^\rho + \alpha_3 M^\rho]^{(1/\rho)-1}(\rho)\alpha_2 S^{\rho-1} \]

and

\[ u_m = (1/\rho)\alpha_0[\alpha_1 G^\rho + \alpha_2 S^\rho + \alpha_3 M^\rho]^{(1/\rho)-1}(\rho)\alpha_3 M^{\rho-1} \]

so

\[
- \frac{dM}{dS} \bigg|_{dG=0} = MRS_{SM} = \frac{u_s}{u_m}
\]

\[
= \frac{(1/\rho)\alpha_0[\alpha_1 G^\rho + \alpha_2 S^\rho + \alpha_3 M^\rho]^{(1/\rho)-1}(\rho)\alpha_2 S^{\rho-1}}{(1/\rho)\alpha_0[\alpha_1 G^\rho + \alpha_2 S^\rho + \alpha_3 M^\rho]^{(1/\rho)-1}(\rho)\alpha_3 M^{\rho-1}}
\]

\[
= \frac{\alpha_2 S^{\rho-1}}{\alpha_3 M^{\rho-1}} = \frac{\alpha_2}{\alpha_3} \left( \frac{S}{M} \right)^{(\rho-1)}
\]

Now assume that \( \alpha_0 = 5, \alpha_1 = 4.5, \alpha_2 = 1.5, \alpha_3 = .5 \) and \( \rho = 1.5 \). How many Big Macs will Gertrude be willing to give up to receive an additional bottle of scotch if she is currently consuming two Big Macs and eighteen bottles of scotch? [Show your math but also answer in words.]

**answer:**

\[ MRS(18, 2)_{SM} = \frac{\alpha_2}{\alpha_3} \left( \frac{18}{2} \right)^{(\rho-1)} = 3(9)^{.5} = 9 \]

So, if Gertrude is consuming 18 bottles of scotch and 2 BigMacs, she will give up 9 Macs to get one more bottle of scotch.

How many Big Macs would she be willing to give up to get one more bottle of scotch if she is currently consuming four Big Macs and one bottle of scotch? [Show your math but also answer in words.]
answer:

\[ MRS(18, 2)_{SM} = \frac{a_2}{a_3} \left( \frac{1}{4} \right)^{(\rho-1)} = 3 \left( \frac{1}{4} \right)^5 = 1.5 \]

That is, if Gertrude is currently consuming 1 bottle of scotch and 4 BigMacs, she will give up 1.5 Big Macs to get one more bottle of scotch. For example, if \( a_3 = .01 \), Gertrude will give up 480 Big Macs to get one more bottle of scotch.

1. Assume that Gertrude the recluse exists on just three goods; grapefruit (\( G \)), scotch (\( S \)), and Big Macs (\( M \)). Her preferences for these three goods can be represented by some utility function:

\[ u = u(G, S, M) \]

Derive, using the concept of a total differential, Gertrude’s marginal rate of substitution between scotch and Big Macs, \( MRS_{SM} \)

\[ \frac{dM}{dS} \frac{du}{dG} = -MRS_{SM} \]

answer: Find the total differential of \( u \), \( du \), set it equal to zero and solve to find

\[ MRS_{SM} = \frac{u_s}{u_m} \]

Now assume that Gertrude’s preferences can be represented by the specific utility function:

\[ u = u(G, S, M) = a_0 [a_1 G^{-\rho} + a_2 S^{-\rho} + a_3 M^{-\rho}]^{-1/\rho} \]

where \(-1 < \rho \neq 0\) and \( a_i > 0 \ \forall i \).

Determine Gertrude’s marginal rate of substitution between scotch and Big Macs, \( MRS_{SM} \).

answer:

\[ u_s = (-1/\rho)a_0[a_1 G^{-\rho} + a_2 S^{-\rho} + a_3 M^{-\rho}]^{-(1/\rho)-1}(-\rho)a_2 S^{-\rho-1} \]

and

\[ u_m = (-1/\rho)a_0[a_1 G^{-\rho} + a_2 S^{-\rho} + a_3 M^{-\rho}]^{-(1/\rho)-1}(-\rho)a_3 M^{-\rho-1} \]

so

\[ - \frac{dM}{dS} \frac{du}{dG} = MRS_{SM} = \frac{u_s}{u_m} = \frac{(-1/\rho)a_0[a_1 G^{-\rho} + a_2 S^{-\rho} + a_3 M^{-\rho}]^{-(1/\rho)-1}(-\rho)a_2 S^{-\rho-1}}{(-1/\rho)a_0[a_1 G^{-\rho} + a_2 S^{-\rho} + a_3 M^{-\rho}]^{-(1/\rho)-1}(-\rho)a_3 M^{-\rho-1}} \]

\[ = \frac{a_2 S^{-\rho-1}}{a_3 M^{-\rho-1}} = \frac{a_2}{a_3} \left( \frac{S}{M} \right)^{(\rho+1)} \]

Now assume that \( a_0 = 5, a_1 = 4.5, a_2 = 1.5 \) and \( \rho = 1.5 \). How many Big
Macs will Gertrude be willing to give up to receive an additional bottle of scotch if she is currently consuming two Big Macs and eighteen bottles of scotch? [Show your math but also answer in words.]

answer: \[ MRS(18, 2)_{SM} = \frac{a_2}{a_3} \left( \frac{18}{2} \right)^{(p+1)} = \frac{1.5}{a_3} (9)^{-(1.5+1)} = \frac{1.5}{a_3} (9)^{-2.5} \]

So, if Gertrude is consuming 18 bottles of scotch and 2 BigMacs, she will give up only \( \frac{0.00617}{a_3} \) Big Macs to get one more bottle of scotch. The specific numerical value depends on the value of \( a_3 \). For example, if \( a_3 = 0.01 \) Gertrude will give up 0.6 Big Macs to get one more bottle of scotch.

How many Big Macs would she be willing to give up to get one more bottle of scotch if she is currently consuming four Big Macs and one bottle of scotch? [Show your math but also answer in words.]

answer: \[ MRS(1, 4)_{SM} = \frac{a_2}{a_3} \left( \frac{1}{4} \right)^{(p+1)} = \frac{1.5}{a_3} (.25)^{-(1.5+1)} = \frac{1.5}{a_3} (.25)^{-2.5} \]

That is, if Gertrude is currently consuming 1 bottle of scotch and 4 BigMacs, she will give up \( \frac{48}{a_3} \) Big Macs to get one more bottle of scotch. For example, if \( a_3 = 0.01 \) Gertrude will give up 480 Big Macs to get one more bottle of scotch.

2. Assume that there are only three commodities in the world; cigars \((c)\), bowling \((b)\) and tofu \((t)\). Further assume that Ralph’s weird preferences for these three commodities can be represented by the utility function

\[ u = u(c, b, t) = cb^{1.5} + 4c^{-5}t^5 \]

where \( c, b, t > 0 \) and \( b \leq 3 \). God does not let him bowl more than 3 times.

Ralph is currently consuming \( c_0 \) cigars, bowling \( b_0 \leq 3 \) games and eating \( t_0 \) lbs of tofu. What is the maximum amount of tofu he will be willing to give up to bowl one more game, assuming he is currently bowling 2 or fewer times.

Make sure you explain both your derivation and your answer in words.

How many lbs of tofu would Ralph be willing to give up to bowl one more game if he is currently smoking 4 cigars, eating 9 lbs of tofu and bowling 1 game?

Does the amount of tofu he is willing to give up to bowl one more game increase or decrease as a function of the amount of tofu he is initially consuming? Use a derivative to answer this question. Make sure to explain your answer in words.

answer: For fun? I will begin by graphing the function holding the level of cigars and utility constant. For example, assume \( u_0 = 5 \), and \( c_0 = 1 \).

\[ 5 = b^{1.5} + 4t^5 \]

Solution is: \( \left\{ t = 1.5625 - 0.625b^{\frac{3}{5}} + 0.0625b^3 \right\} \)

\[ 562.5 - 0.625b_1^{\frac{3}{5}} + 0.0625b_1^3 \]
indifference curve: c=1, u=5
now graph it for \( b \) up to 6
\[
1.5625 - 0.625b^{1/3} + 0.0625b^3
\]
The indifference is negatively sloped for games bowled less than 3 then it goes weird.

Given that \( u = u(c, b, t) = cb^{1.5} + 4c^5t^5 \). The maximum amount of tofu he is willing to give up to bowl one more game is

\[
\frac{dt}{db} \bigg|_{du=0} = -\frac{\frac{\partial u}{\partial b}}{\frac{\partial u}{\partial t}} = -\frac{1.5cb^{-5}}{2c^5t^{-5}} = -\frac{1.75c^{-2}b^{-5}t^{-5}}{0 \forall c, b, t > 0}
\]

Evaluate this when \( c = 4, t = 9, \) and \( b = 1 \), \(-.75(4)^{-2}1^{-5}9^{-5} = -4.5\). Ralph will give up 4.5 lbs of tofu to bowl one more game.

Note that when \( c = 4, t = 9, \) and \( b = 1 \), utility is \( 4 + 8(3) = 28 \). Now let’s graph the indifference curve when \( c^0 = 4 \), and utility is 28. That is, \( 28 = 4b^{1.5} + 8t^5 \), Solution is:
\[
\{t = 12.25 - 3.5b^{1/3} + 0.25b^3 \} \quad t = 12.25 - 3.5b^{1/3} + 0.25b^3
\]
Indifference curve: $c=4, u=28$

Note from the graph that we can see that tofu consumption has to drop from 9 to 4.5 when bowling increases from 1 to 2 games.

To determine whether the amount of tofu he is willing to give up to bowl one more game ($\frac{dt}{dc} \bigg|_{du=0} = -0.75c^{-5}b^{-5}t^{-5}$) increases or decreases as his initial consumption of tofu increases, one needs to take the partial derivative of ($\frac{dt}{db} \bigg|_{du=0} = -0.75c^{-5}b^{-5}t^{-5}$) with respect to $t$. That is

$$\frac{\partial}{\partial t} (-0.75c^{-5}b^{-5}t^{-5}) = -0.375 \sqrt{c} \frac{\sqrt{b}}{\sqrt{t}}$$

which is negative. That is, the amount of tofu he is willing to give up to bowl one more game increases as initial tofu consumption increases. In terms of the above indifference curve, its slope becomes steeper as $t$ increases. That is, the more tofu he starts with the more he is willing to give up to bowl one more game. Think of out the other way. What happens to $\frac{dt}{db} \bigg|_{du=0} = -0.75c^{-5}b^{-5}t^{-5}$, as $b$ increases? $\frac{\partial}{\partial b} (-0.75c^{-5}b^{-5}t^{-5}) = -0.375 \frac{\sqrt{c}}{\sqrt{t}} \sqrt{t} < 0$. **Aaron, what gives?** How can the slope become steeper now matter which direction you go.

To see how the slope changes lets, look at the above indifference curve but zoom in to the area around $b = 1$
Indifference curve: c=4, u=28

It is almost a straight line. At \( b = c = 1 \), \( \frac{\partial}{\partial t} (-0.75 t^{-5}) = -0.375 \frac{1}{t^6} \). Evaluated at \( t = 9 \), 
\(-0.375 \frac{1}{9^6} = -0.125\). the slope of the indifference curve is decreasing, but at a very slow rate.

1. Given the utility function

\[ u = u(x_1, x_2) \]

Derive the slope of the individual’s indifference curve; that is, derive

\[ \frac{dx_2}{dx_1} \bigg|_{du=0} = -MRS_{x_1,x_2} \]

What is the \( MRS_{x_1,x_2} \) if

\[ u = u(x_1, x_2) = a_1 x_1^\beta + a_2 x_2^\beta \]

What is the specific value of \( MRS_{x_1,x_2} \) if \( a_1 = 2, a_2 = 1, \beta = .5, x_1 = 20 \) and \( x_2 = 5 \)? What does the answer tell you about preferences?

2. Be able to define utility functions, indifference curves, production functions and isoquants. Know how to derive the slope of the indifference (isoquant) curve from the utility (production) function using the concept of the total differential.

3. Assume a world of two goods, \( x_1 \) and \( x_2 \). Parts 1 and 2: Define in both words and functional notation, an individual’s budget line. Parts 3 and 4: Define in words and in set notation an individual’s indifference curve for \( u = u^0 \). Part 5 and 6: Define in both words and functional notation the rate at which the market allows the individual to substitute \( x_1 \) for \( x_2 \). Part 7 and 8: Define in both words and functional notation the rate at which the individual is willing to substitute \( x_1 \) for \( x_2 \). Part 9: In words, what is the difference between these two rates

answer: The budget line is

\[ y = p_1 x_1 + p_2 x_2 \]

where \( y \) is exogenous income and \( p_i \) is the exogenous price of commodity \( i \). In words, the budget line identifies all those combination of \( x_1 \) and \( x_2 \) that the individual can just afford given his income and the prices. An individuals indifference curve for the utility level \( u^o \) is all those combinations of \( x_1 \) and \( x_2 \) that just achieve utility level \( u^o \). In terms of set theory, it is
\{(x_1, x_2) : u(x_1, x_2) = u^o\}

The rates at which the market allows the individual to substitute \(x_1\) for \(x_2\) is

\[
\frac{dx_2}{dx_1} \bigg|_{du=0}
\]

In words, it is how much the market forces the individual to reduce her consumption of \(x_2\) if she increases her consumption of \(x_1\) by one unit. The rate at which individual is willing to substitute \(x_1\) for \(x_2\) is

\[
\frac{dx_2}{dx_1} \bigg|_{dw=0}
\]

In words, it is how much \(x_2\) would have to decrease if \(x_1\) increases by one unit and the individual remains indifferent. Said another way, it is how much \(x_2\) the individual would give up to get one more unit of \(x_1\).

\[
\frac{dx_2}{dx_1} \bigg|_{du=0}
\]

is determined by the individual’s constraints (the prices of the two goods) and

\[
\frac{dx_2}{dx_1} \bigg|_{dw=0}
\]

is determined by the individual’s preferences.

4. Assume some function, \(z = g(x_1, x_2, x_3)\). Define both mathematically, and in words, the differential \(dz\).

5. Define, in words, the production function \(y = f(k, l), k > 0, l > 0\).

Define, in words, the isoquant associated with \(y_0\) units of output.

Using the concept of a total differential, determine \(\frac{dk}{dl} \bigg|_{dy=0}\). Show all your work and explain, in words each step of your derivation.

Now assume

\[y = f(k, l) = a_0 + a_1 l + a_2 k\]

What is the \(MRTS_{lk}\) when \(y = 5 + .7l + .7k\), and \(l = 2\) and \(k = 4\).

Given \(y = 5 + .7l + .7k\), does the magnitude of \(MRTS_{lk}\) depend on the magnitudes of \(l\) and \(k\)? Given \(y = 5 + .7l + .7k\), how would you define the relationship between the labor and capital inputs? Would this relationship hold if \(a_1 = .7\) and \(a_2 = .1\)?

**answer:** We have derived the general result that

\[
\frac{dk}{dl} \bigg|_{dy=0} = -\frac{f_l}{f_k}
\]

In this case, \(f_l = a_1\) and \(f_k = a_2\), so \(\frac{dk}{dl} \bigg|_{dy=0} = -\frac{a_1}{a_2}\). So the \(MRTS_{lk} = \frac{a_1}{a_2}\) and does not depend on the quantities of labor or capital. If \(a_1 = a_2 = .7\), it is 1. In words, labor and capital substitute for one another at the constant rate of 1 for 1. They are perfect substitutes. It would still hold.

6. Assume a firm that produces product \(x\) using labor and capital. Its production function is

\[x = f(k, l)\]

Define in words, and in functional notation, the firm’s marginal rate of technical substitution of labor for capital. Now assume
\[ x = f(k, l) = 3 \ln k + 2 \ln l \]

What is this specific firm’s marginal rate of substitution of labor for capital?
What is the marginal rate of substitution of labor for capital expressed in percentage terms.

**Answer:** The marginal rate of technical substitution of labor for capital is

\[ \text{mrts}_{lk} = \frac{dk}{dl} \bigg|_{dx=0} \]

This is determined as follows.

\[ dx = f_idx + fkd_l \]

which along the isoquant equals zero. So, along the isoquant

\[ f_idx + fkd_l = 0 \Rightarrow \frac{dk}{dl} \bigg|_{dx=0} = -\frac{f_i}{f_k} \]

and

\[ \text{mrts}_{lk} = \frac{f_i}{f_k} \]

This is true no matter what the production function. In this particular case,

\[ f_i = 2l^{-1} \]
\[ f_k = 3k^{-1} \]

So

\[ \text{mrts}_{lk} = \frac{f_i}{f_k} = \frac{2l^{-1}}{3k^{-1}} = \frac{2k}{3l} \]

Expressing this in percentage terms

\[ -\left[ \frac{\Delta k}{\Delta l/l} \bigg|_{dx=0} \right] = -\left[ \frac{\Delta k}{\Delta l} \bigg|_{dx=0} \right] l/k = \frac{f_i}{f_k} \cdot \frac{1}{k} = \frac{2k}{3l} \cdot \frac{l}{k} = \frac{2}{3} \]

That is, a one percent increase in \( l \) requires a 2/3% decrease in \( k \) if output is to remain constant.

1. Define, in words, the utility function \( u(x_1, x_2), x_1, x_2 > 0 \).
Define, in words, the indifference curve associated with \( u^o \) units of utility.
Using the concept of a total differential, determine \( \frac{dx_1}{dx_2} \bigg|_{dx=0} \). Show all your work and explain, in words each step of your derivation.
Now assume

\[ u = u(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2^\gamma \]

What is the \( MRS_{x_2x_1} \) when \( u = 5 + 4x_1 + 8x_2^5 \)?
What does this mean when \( x_2^o = 9 \)?

**answer:**
The utility function asssociates a number with each bundle of the two goods, such that if the individual is indifferent between two bundles the function
assigns the same number to each, and if a bundle is preferred to another
bundle, the function assigns a larger number to the preferred bundle.
The indifference curve associated with the utility level \( u^0 \) consists of all those
bundles of the two goods that just achieve the utility level \( u^0 \)
The total differential of the utility function is
\[
du = \frac{\partial u}{\partial x_1} \, dx_1 + \frac{\partial u}{\partial x_2} \, dx_2
\]
along an indifference curve, utility does not change, so
\[
0 = \frac{\partial u}{\partial x_1} \, dx_1 + \frac{\partial u}{\partial x_2} \, dx_2
\]
Solve this to get
\[
\frac{dx_1}{dx_2} \bigg|_{du=0} = -\frac{\frac{\partial u}{\partial x_2}}{\frac{\partial u}{\partial x_1}}
\]
Given \( u(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2^y \)
\[
\frac{dx_1}{dx_2} \bigg|_{du=0} = -\frac{\frac{\partial u}{\partial x_2}}{\frac{\partial u}{\partial x_1}} = -\frac{\gamma a_2 x_2^{y-1}}{a_1}
\]
When \( u = 5 + 4x_1 + 8x_2^5 \)
\[
\frac{dx_1}{dx_2} \bigg|_{du=0} = -\frac{\gamma a_2 x_2^{y-1}}{a_1} = -\frac{5(8)x_2^{-5}}{4} = -x_2^{-5}
\]
When \( x_2 \) is increases by one unit \( x_1 \) must decrease by \( x_2^{-5} \) units.
\( MRS_{x_2x_1} = x_2^{-5} \)
Evaluate this when \( x_2^5 = 9, 9^{-5} = 0.33333 \). This means that when the
individual is consuming 9 units of \( x_2 \), if \( x_2 \) is increased by 1 unit the
consumption of \( x_1 \) must fall by 1/3 units to keep utility constant.

1. Define, both in words and in mathematical notation, utility functions,
indifference curves, production functions and isoquants. Derive the slope of
an indifference curve from a utility function using the concept of the total
differential. Explain your steps in words.

answer:
The utility function associates a number with each bundle of goods such
that if the individual is indifferent between two bundles the function assigns
the same number to each bundle, and if a bundle is preferred to another
bundle, the function assigns a larger number to the preferred bundle. In
mathematical notation, \( u = u(x_1, \ldots, x_N) \) where \( x_i \) is the quantity consumed of
good \( i \). An individual’s utility function describes the individual’s preferences
over bundles of goods.
The indifference curve associated with the utility level $u^0$ consists of all those bundles of the goods that just achieve the utility level $u^0$. In mathematical notation, $I(u^0) \equiv \{(x_1,\ldots,x_N) : u(x_1,\ldots,x_N) = u^0\}$. Another acceptable answer is the indifference curve associated with bundle $x^0 = (x_1^0,\ldots,x_N^0)$ is all those bundles such that the individual is indifferent between each of them and each indifferent with the bundle $x^0$. In mathematical notation, $I(x^0) \equiv \{(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \sim x^0\}$. Put simply, an individual is indifferent between all bundles on the same indifference curve.

The total differential of the utility function is

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_N} dx_N$$

along an indifference curve, utility does not change, so

$$0 = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \ldots + \frac{\partial u}{\partial x_N} dx_N$$

To get the slope of this indifference curve in the $x_i x_j$ direction set $dx_k = 0$ for all $k \neq j$ or $i$ to obtain

$$0 = \frac{\partial u}{\partial x_j} dx_j + \frac{\partial u}{\partial x_i} dx_i$$

Solve this to get

$$\frac{dx_j}{dx_i} \bigg|_{dx_i=0, dx_k=0} = -\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}}$$

which is the slope of the indifference curve. Put simply, it is how much $x_j$ must change to hold utility constant if $x_i$ is increased by one unit.

A production function identifies maximum output as function of the quantities used of the different inputs. In mathematical notation, $y = f(x_1,\ldots,x_N)$ where $y$ is units of output and $x_i$ is the quantity used on input $i$. The production function describes the state of technical knowledge for production commodity $y$. The isoquant associated with output level $y^0$ consists of all those input combinations that are just capable of producing $y^0$. In mathematical notation, $I(y^0) \equiv \{(x_1,\ldots,x_N) : f(x_1,\ldots,x_N) = y^0\}$. Put simply, an isoquant identifies all input combination that are all just capable of producing the same maximum output level.