1 Notes on congestion and road tolls

Edward Morey November 17, 2016

Some things to think about to get us started.

Tolls should ideally vary by time of day and day of the week.

If tolls vary by time of day and day of week, they could actually increase the number if drivers on the road and make everyone better off (Henderson 1974). This is an interesting result, most people assume a road toll will reduce the total number of drivers, but it doesn’t have to if the toll is varying.

There is a relationship between speed, flow and the number of cars on the road. This relationship varies by weather (rain, snow, clear) and light (night, dusk, etc.).

2 A stylized road example: Pigou’s road to Breck

Assume there are two roads from the front range to Breckenridge: a very wide road and a narrow road, I-70.

Assume 10,000 skiers drive to Breck every Saturday morning (they all have ski passes and feel compelled to go), and they all drive alone (they are paranoid about having others in the car).\(^1\)

Driving time to Breck on the wide road is always 2.5 hours (150 minutes) no matter how many cars are on the wide road - 10,000 cars would not congest it; it is a wide, magical road.

In contrast, the narrow road takes approximately 44 minutes if only one car takes it, but marginal travel time increases as the number of cars on the narrow road increases (how much total travel time changes), The road is congestible.

\(^1\)Assuming a fixed number of drives is unrealistic is simplifies the problem.
Specifically assume $mc_n(k_n) = 44 + .2k_n$ where $k_n$ is the number of cars on the narrow road.\footnote{The CDOT had a bunch of traffic engineers do a study. They observe average speed with different traffic loads and used the data to estimate this marginal cost function, in terms of time.} For example, if they are 100 cars on the narrow road adding another increases total travel time of the narrow road by $44 + .2(100) = 64$ minutes.

Note that by assumption $k_n + k_w = 10,000$. We assumed away the problem of determining the number of people who will want to go to Breck by assuming it fixed at 10,000.

Given these assumptions total travel time for all of the cars on the narrow road is $tc_n(k_n) = 44k_n+.1k_n^2$ and total travel time on the wide road is $tc_w(k_w) = 150k_w$.\footnote{How do I get $tc(k_n)$ from $mc(k_n)$? I integrated the marginal cost curve wrt $k_n$.} It follows that $ac_w = 150$ and $ac_n(k_n) = \frac{44k_n+.1k_n^2}{k_n} = 44 + .1k_n$. Marginal cost and average cost for the narrow road are the two upward sloping lines. (note that the time it takes you to drive the narrow road is $ac_n(k_n)$, not $mc_n(k_n)$.)\footnote{So, if you are the 100th car on the narrow road it will take everyone on the narrow road $ac_n(100) = 44 + .1(100) = 54$ minutes. But note that total travel time by everyone increases by 64 minutes.}

If everyone has the same value of time, e.g. $\$10$ hour, these costs can also be easily expressed in dollars.
So, what does all of the above say? For the wide road marginal cost equals average cost equals 150 minutes.

For the narrow road, marginal cost is greater than average costs so as the number of cars on the road increases, average driving time increases. For example if, there are currently 1000 cars on the narrow road, marginal cost is 244 minutes (the amount total travel time increases if one more car takes the road), but average cost is only 144 minutes.

Average cost is how long it takes each driver to make it to Breck if there are 1000 drivers on the road. That is, the last guy who decides to take the road will take 144 minutes to get to Breck (same time as everyone else) but total travel time goes up not by 144 but rather by 244. Why? The last guy’s presence on the road slows everyone else down by .1 minutes (6 seconds)–this is the increase in average time.
When a driver decides which road to take on Saturday morning, she compares 150 with how long it will take her on the narrow road, which is average cost (how long it will take her). So if

- if $150 > 44 + .1k_n$ take the narrow road
- if $150 < 44 + .1k_n$ take the wide road
- if $150 = 44 + .1k_n$ indifferent

So, in equilibrium (when no one wants to switch roads), $150 = 44 + .1k_n$. Solving, $k_n = 1060$. That is, if both roads are common-property resources (access is uncontrolled), 1060 cars will take the narrow road, and 8940 will take the wide road.

Total travel time for the 10,000 drivers will be

$$tc(1060) = 44(1060) + .1(1060)^2 + 150(10000 - 1060)$$
$$= 44(1060) + 1060^2 + 150(9940)$$
$$= 44(1060) + 1560000 + 150(9940)$$
$$= 46640 + 1560000 + 1491000$$
$$= 3048400$$

$3048400$ minutes $= 25,000$ hours $= $250,000
Is this the efficient allocation of the 10,000 cars between the two roads?

No.

Why not?

Total travel cost is not being minimized because there is a wedge between the cost to a driver of taking the narrow road \( ac_n(k_n) = 44 + .1k_n \) and cost to society of her taking the narrow road \( mc_n(k_n) = 44 + .2k_n \). The wedge is \(.1k_n\).

What allocation of the cars would minimize total travel time to Breck? Find the \( k_n \) that minimizes

\[
\text{tc}(k_n) = 44(k_n) + .1(k_n)^2 + 150(10000 - k_n) = 0.1k_n^2 - 106k_n + 150000
\]

How do we find this time minimizing \( k_n \), \( k^*_n \). First find the derivative of \( tc(k_n) \) wrt \( k_n \).

\[
\frac{d\text{tc}(k_n)}{dk_n} = 44 + .2k_n - 150 = 0.2k_n - 106
\]

\( k^*_n \) is the \( k_n \) for which this derivative is zero. Solving \( 0.2k_n - 106 = 0 \), Solution is: 530.0. Wow - could this be correct it says 530 cars should be on the narrow road and 10000 - 530 = 9470 on the wide road.
Let’s check our answer another way. When the efficient number of cars are on the narrow road the social cost of additional car driving to Breck on the narrow road should equal the social cost of that additional car driving to Breck on the wide road, which is 150 minutes.

Mathematically, marginal cost on narrow road equal marginal cost on wide road when

\[ 150 = 44 + .2k_n \]

Solution is: 530.0, confirming our earlier answer.

\[
44(1060) + .1(1060)^2 + 150(10000 - 1060) - (44(530) + .1(530)^2 + 150(10000 - 530)) = 28090.0
\]

So, how much time is wasted by the misallocation of cars when access is not controlled. Total travel time with the efficient allocation is

\[ tc(530) = 44(530) + .1(530)^2 + 150(10000 - 530) \]
\[ = 1.4719 \times 10^6 = 1,471,900 \text{ minutes} \]
\[ = 24,532 \text{ hours} = $245,320 \]

The difference is \( 1.5 \times 10^6 - 1.4719 \times 10^6 = 28100 \text{ minutes} = 28100/60 = 468.33 \text{ hours} = $4680 \) wasted because the cars were inefficiently allocated between the two roads.
How to fix the problem?

Close the narrow road when the 530th car gets on. If this is the solution, 468.33 hours of driving are saved, time that could be used to do other stuff like sleeping or doing one’s homework. Some drivers will be made better off.

Will anyone be made worse off (experience increased driving time)? In the common-property equilibrium driving to Breck takes 150 minutes on either road, when the the narrow road is closed after the 530th car, drivers on the wide road still take 150 minutes and those on the narrow road take $44 + .1(530) = 97$ minutes. So, no driver takes more time and 530 drivers take 63 minutes less - definitely a Pareto Improvement.

One problem with this scheme is it might cause a race to I-70. One would want to have a reservation system, register your car online for access. You get fined if you do not show up.
Or, efficiency could be achieved by charging a toll on the narrow road. What should the toll be? We want to set the toll so that equilibrium is where 530 cars choose to take the narrow road.

First express the toll in minutes, which we will then convert to dollars. In equilibrium, we want marginal private costs of taking the narrow road, including the toll, to equal 150 when there are 530 cars on the narrow road. Let $mc_w$ be the marginal cost on the narrow road including the toll.

$$mc_w = 150 = 44 + .1(530) + toll_m$$

Solution is: $toll_m$ equals 53 minutes. That is, the toll should be a wait of 53 minutes where during the wait the driver works for the good of mankind or doesn’t wait but pays a dollar toll of $(53/60)10 = \$8.83$.

So are the drivers better off or worse off because we charged the toll. Total cost to the drivers with no toll is $\$250,000$. With the $\$8.83$ toll it is

$$tc = \$245,320 + \$8.83(530)$$

$$= \$250,000$$

Wow - the total cost to the 10,000 drivers is the same whether the allocation is efficient or common property. So, the drivers are, not worse off because of the toll, either individually or as a group. The drivers who paid the toll traded money for time. The money is now available to make others, or even the drivers, better off. It could be used to feed poor kids lunch at school (over a 1000 a week) or even used to pay for improved roads. Of course, if I-70 was widened, the efficient toll would change.
Now, let’s ask another question. What if I-70 was managed to maximize revenues from the toll?

First we need to determine the number of cars that will take the narrow road as a function of the toll.

We know in equilibrium, \( 150 = 44 + .1(k_n) + toll \); solving \( k_n = k_n(toll) = 1060.0 - 10.0toll \), which is the demand function for trips on the narrow road.

So, revenue from the toll is toll times demand as a function of the toll\(^5\)

\[
R(t) = (1060.0 - 10.0t)t = 1060t - 10t^2
\]

At what \( t \) is the revenue from the toll maximized. Graphing it \( 1060t - 10t^2 \)

\(^5\)For brevity let \( t \) denote \( toll \).
Notice where this is maximized, at 53 minutes, the same answer we got when we choose the toll to minimize total travel time by the 10,000 cars.

Wow - a private owner maximizing revenues\textsuperscript{6} would achieve the efficient allocation of cars between the two roads. Adam Smith’s invisible cruise control

\textsuperscript{6}maximizing revenues would maximize profits if maintenance was not a function of the number of cars
This road example is obviously stylized and restrictive - but it gives the flavor of things. Many road are highly congested from an efficiency point of view.

Restrictive assumptions included the existence of a non-congestible substitute and the assumption that the number of Saturday morning skiers would not be affected by the toll or access restriction: a toll that varied by time of day and day of week would cause substitution away from driving I-70 on Saturday morning, substitution to other days and times, and substitution to sleeping and watching football.

The writings of Toby Page motivated these notes.
Consider the following data from


It indicates that Westbond I-70 on Saturdays in the winter at Idaho Springs has a volume of almost 4000 cars per hour at 7 a.m., official capacity is about 3000 cars per hour. Eastbound reaches its peak on Saturday and Sunday afternoon at around 4 p.m. with a volume of about 3500 cars per hour.

Joy
Thank you so much.
I will send you what I write up, when I finish it.
Thanks again.
Edward

---

From: Joy, Cecelia [mailto:Cecelia.Joy@dot.state.co.us]
Sent: Monday, October 17, 2005 4:37 PM
To: Edward Morey
Cc: Paulsen, Chris
Subject: RE: I-70 corridor - research question.
Ed, I’m not sure of the “June 200?) PEIS report that you reference. The most current source of information is contained in the I-70 PEIS, Vol 1 and 2 dated December 04. The data you are interested in is provided in Appendix b of Volume 2. See www.i70mtncorridor.com. Let me or Chris Paulsen (the project manager) know if you have any other questions.

---

From: Edward Morey [mailto:Edward.Morey@Colorado.edu]
Sent: Sunday, October 16, 2005 4:25 PM
To: Joy, Cecelia
Subject: I-70 corridor - research question.
Joy Cecelia
I am an economist at C.U. who has an interest in transportation economics.
I am wondering if there is a study or estimate of the relationship between travel times and traffic volume on I-70 west of Denver. For example, how long it takes to drive from Golden to the tunnel as a function of traffic volume. I am particularly interested in winter travel between Denver and the tunnel, but anything related would be great. Thanks.
Any information you might have would be greatly appreciated.
I already have a copy of the June 200 PEIS report
Thanks
Edward
Edward Morey
Professor of Economics
Department of Economics
Campus Box 256
University of Colorado
Boulder, Colorado 80309-0256
Edward.Morey@Colorado.edu
http://www.colorado.edu/Economics/morey/