A Primer on Difference Equations

1. First-Order Difference Equations

Consider the following differential equation:

\[ y_{t+1} + \beta y_t = \kappa, \quad (1.1) \]

where \( y \) is a variable of interest, \( t \) is time, and \( \beta \) and \( \kappa \) are constants. The solution to this differential equation is the sum of a particular integral \( y^p \) and a complementary function \( y^c \):

\[ y_t = y^p + y^c. \quad (1.2) \]

The solution depends on the value of \( \beta \).

The Particular Integral

The particular integral is any solution to the full nonhomogenous equation. For example, assume that \( y \) is constant over time. Then, \( y_{t+1} = y_t \) and

\[ y^p = \kappa/(1 + \beta) \quad (\beta \neq -1). \quad (1.3) \]

The Complementary Function

The complementary function is the solution to the homogenous equation

\[ y_{t+1} + \beta y_t = 0, \quad (1.4) \]

We guess that the solution is of the form

\[ y^c = \Gamma \rho^t, \quad (1.5) \]

where \( \Gamma \) and \( \rho \) are an undetermined coefficients. The value of \( \rho \) is found as follows. Note that if our guess holds, \( y_{t+1} = \Gamma \rho^{t+1} \), and we can write (1.4) as:

\[ \Gamma \rho^{t+1} + \beta \Gamma \rho^t = 0, \quad (1.6) \]

which implies that \( \rho = -\beta \).

The Full Solution
To obtain the full solution, we must identify $\Gamma$. As long as $\beta \neq 0$, we have:

\[ y_t = y^p + y^c, \]

\[ = \frac{\kappa}{(1 + \beta)} + \Gamma (-\beta)^t. \quad (1.7) \]

We find $\Gamma$ by imposing an initial condition. In this case, we assume that $y_0 = \bar{y}$ at time $t = 0$. Also, at time $t = 0$, we have $y_0 = \frac{\kappa}{(1 + \beta)} + \Gamma(-\beta)^0$, such that $\Gamma = \bar{y} - \kappa/(1 + \beta)$.

The full solution is then

\[ y_t = \frac{\kappa}{(1 + \beta)} + \left( \bar{y} - \frac{\kappa}{(1 + \beta)} \right) (-\beta)^t \quad (\beta \neq -1) \quad (1.8) \]

In the case where $\beta = -1$, the differential equation reduces to

\[ y_{t+1} - y_t = \kappa. \quad (1.9) \]

In this case, the solution is found by straight integration to be

\[ y_t = \kappa t + \Gamma, \quad (1.10) \]

where $\Gamma$ is an undetermined coefficient. Using our initial condition, $y_0 = \bar{y} = \kappa 0 + \Gamma$, such that $\Gamma = \bar{y}$. The full solution is

\[ y_t = \bar{y} + \kappa t \quad (\beta = -1). \quad (1.11) \]

So, the solutions are:

\[ y_t = \begin{cases} \frac{\kappa}{(1 + \beta)} + [\bar{y} - \kappa/(1 + \beta)] (-\beta)^t & \text{if } \beta \neq -1 \\ \bar{y} + \kappa t & \text{if } \beta = -1. \end{cases} \quad (1.12) \]

2. Second-Order Differential Equations

The differential equation can be written as:

\[ y_{t+2} + \alpha y_{t+1} + \beta y_t = \kappa, \quad (2.1) \]

where $y$ is a variable, $t$ is time, and $\alpha$, $\beta$, and $\kappa$ are constants.

The Particular Integral
The particular integral is any solution to the full nonhomogenous equation. For example, assume that $y$ is constant over time. Then, $y_{t+2} = y_{t+1} = y_t$ and

$$y^p = \frac{\kappa}{1 + \alpha + \beta} \quad (\alpha + \beta \neq -1). \quad (2.2)$$

If $\alpha + \beta = -1$, then assume that $y_t = \eta t$. This implies that $y_{t+2} = \eta(t+2)$, $y_{t+1} = \eta(t+1)$, and (at $t = 0$) $(2 + \alpha)\eta = \kappa$. Thus, $\eta = \kappa/(\alpha + 2)$, and the particular integral is

$$y^p = \frac{\kappa}{(\alpha + 2)} \cdot \quad (\alpha + \beta = -1, \alpha \neq -2). \quad (2.3)$$

Finally, if $\alpha + \beta = -1$ and $\alpha = -2$, we try $y_t = \eta t^2$. It implies that $y_{t+2} = \eta(t + 2)^2$, $y_{t+1} = \eta(t + 1)$, and (at $t = 0$) $(4 + \alpha)\eta = \kappa$. Then, $\eta = \kappa/2$ and

$$y^p = \frac{\kappa}{2} \cdot \quad (\alpha + \beta = -1, \alpha = -2). \quad (2.4)$$

So, the particular integral is:

$$y^p = \begin{cases} 
\frac{\kappa}{1 + \alpha + \beta} & \text{if } \alpha + \beta \neq -1 \\
\frac{\kappa}{(\alpha + 2)} \cdot t & \text{if } \alpha + \beta = -1 \text{ and } \alpha \neq -2 \\
\frac{\kappa}{2} \cdot t^2 & \text{if } \alpha + \beta = -1 \text{ and } \alpha = -2 
\end{cases} \quad (2.5)$$

The Complementary Function

The complementary function is the solution to the homogenous equation

$$y_{t+2} + \alpha y_{t+1} + \beta y_t = 0. \quad (2.6)$$

We guess that the solution is of the form

$$y^c = \Gamma \rho^t. \quad (2.7)$$

This solution implies that $y_{t+2} = \Gamma \rho^{t+2}$ and $y_{t+1} = \Gamma \rho^{t+1}$. Substituting in (2.5), we find

$$\Gamma \rho^{t+2} + \alpha \Gamma \rho^{t+1} + \beta \Gamma \rho^t = 0. \quad (2.8)$$

The solution to this requires to solve the quadratic form:

$$\rho^2 + \alpha \rho + \beta = 0. \quad (2.9)$$

The solution to the quadratic form is:

$$\rho_1, \rho_2 = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}. \quad (2.10)$$
This implies that
\[ \rho_1 + \rho_2 = -\alpha \]
\[ \rho_1 \rho_2 = \beta \]

There are three possible cases.

Case 1: Distinct Real Roots \( \alpha^2 > 4\beta \). In this case, the solution is
\[ y^c = \Gamma_1 \rho_1^t + \Gamma_2 \rho_2^t. \] (2.11)
where \( \rho_1 + \rho_2 = -\alpha \) and \( \rho_1 \rho_2 = \beta \).

Case 2: Repeated Real Roots \( \alpha^2 = 4\beta \). In this case,
\[ y^c = \Gamma_1 \rho^t + \Gamma_2 t \rho^t, \] (2.12)
and \( \rho = -\alpha/2 \).

Case 3: Complex Roots \( \alpha^2 < 4\beta \).
\[ y^c = \Gamma_1 \rho_1^t + \Gamma_2 \rho_2^t, \] (2.13)
where \( \rho_1 = h + vi \) and \( \rho_2 = h - vi \), for \( h = -\alpha/2 \), \( v = \left( \sqrt{4\beta - \alpha^2} \right)/2 \), and \( i = \sqrt{-1} \). We can rewrite equation (2.13) as:
\[ y^c = R^t \left( \Gamma_3 \cos \theta t + \Gamma_4 \sin \theta t \right), \] (2.14)
where \( \Gamma_3 = \Gamma_1 + \Gamma_2 \), \( \Gamma_4 = (\Gamma_1 - \Gamma_2) i \), \( R = \sqrt{\beta^2} \), and \( \theta \) defined such that \( \cos \theta = h/R \) and \( \sin \theta = v/R \).

The Full Solution
The full solution requires to identify \( \Gamma \). Consider first the distinct real root case \( (\alpha^2 > 4\beta) \). In this case \( y = y^p + y^c \) and
\[ y_t = y^p + \Gamma_1 \rho_1^t + \Gamma_2 \rho_2^t, \] (2.15)
where \( \rho_1 + \rho_2 = -\alpha \), \( \rho_1 \rho_2 = \beta \), and \( y^p \) defined in equation (2.5). We require two conditions to find \( \Gamma_1 \) and \( \Gamma_2 \). For example, we impose \( y_0 = \bar{y} \) and \( y_1 = \dot{y} \). Then,
\[ y_0 = \bar{y} = y^p + \Gamma_1 + \Gamma_2 \]
\[ y_1 = \dot{y} = y^p + \rho_1 \Gamma_1 + \rho_2 \Gamma_2. \]
These two equations (with two unknowns) can be solved for \( \Gamma_1 \) and \( \Gamma_2 \).
In the repeated real root case ($\alpha^2 = 4\beta$), we find

$$y_t = y^p + \Gamma_1 \rho^t + \Gamma_2 t \rho^t, \quad (2.16)$$

where $\rho = -\alpha/2$ and $y^p$ defined in equation (2.5). We impose $y_0 = \bar{y}$ and $y_1 = \hat{y}$ to obtain:

$$y_0 = \bar{y} = y^p + \Gamma_1$$

$$y_1 = \hat{y} = y^p + \rho_1 \Gamma_1 + \rho_2 \Gamma_2.$$

These can be solved for $\Gamma_1$ and $\Gamma_2$.

Finally, in the complex root case, $\alpha^2 < 4\beta$, we find

$$y_t = y^p + R^t (\Gamma_3 \cos \theta t + \Gamma_4 \sin \theta t), \quad (2.17)$$

where $h = -\alpha/2$, $v = \left(\sqrt{4\beta - \alpha^2}\right)/2$, $R = \sqrt{\beta^2}$, and $\theta$ defined such that $\cos \theta = h/R$ and $\sin \theta = v/R$. We use $y_0 = \bar{y}$ and $y_1 = \hat{y}$ to obtain:

$$y_0 = \bar{y} = y^p + \Gamma_3,$$

$$y_1 = \hat{y} = y^p + R \Gamma_3 \cos \theta + R \Gamma_4 \sin \theta,$$

since $\cos 0 = 1$ and $\sin 0 = 0$. The two equations can be solved for $\Gamma_3$ and $\Gamma_4$.

References