Several Coordinate Systems are used extensively in Orbit Mechanics. They will be reviewed here.

1. Earth Centered Interval (ECI) and Earth Centered Fixed (ECF)

The ECI frame has its origin at the center of mass of the Earth but has a fixed inertial direction along the intersection of the Earth equatorial plane and the ecliptic plane. Although this frame is referred to as inertial it is actually only pseudo inertial because the center of mass of the Earth accelerates due to perturbations from the Moon and other planets. The ECF frame has the same origin but is fixed in the Earth with its \( X' \)-axis through the Greenwich meridian (zero longitude).

\[
\begin{align*}
\theta_g & \equiv \text{Greenwich Sidereal Time} \\
\theta_{g_0} & = \theta_g + \omega (t - t_0) \\
\theta_{g_0} & \equiv \text{Greenwich hour angle} \\
& \quad \text{(Greenwich Sidereal Time) at the epoch, } t_0.
\end{align*}
\]

The transformation matrix between these frames is

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix}_E =
\begin{bmatrix}
C_{\theta_g} & -S_{\theta_g} & 0 \\
S_{\theta_g} & C_{\theta_g} & 0 \\
0 & 0 & 1
\end{bmatrix}_E
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}_E. \tag{2}
\]
2. Another useful frame is the P,Q,W frame

where \( \overrightarrow{P} \) and \( \overrightarrow{Q} \) are unit vectors in the orbit plane with \( \overrightarrow{P} \) directed to perigee, 
\( \overrightarrow{W} \) along the angular momentum vector and \( \overrightarrow{Q} \) completing a right hand triad, 
i.e., \( \overrightarrow{P} \times \overrightarrow{Q} = \overrightarrow{W} \)

To go from XYZ (ECI) to P,Q,W 
We first rotate about Z through \( \Omega \), the right ascension of the ascending node. Next rotate about the new X-axis through I, the inclination, then rotate about the resulting Z-axis through \( \omega \), the argument of perigee.

Here \( P, Q, W \) are the components of a vector expressed in the PQW frame, i.e.,

\[
[\vec{r}]_{PQW} = P\overrightarrow{P} + Q\overrightarrow{Q} + W\overrightarrow{W}.
\]  

(3)

Remember \( \overrightarrow{P}, \overrightarrow{Q}, \overrightarrow{W} \) are unit vectors. These three rotations will yield the desired transformation matrix as the product of three matrices, i.e.,

\[
\begin{bmatrix}
P \\
Q \\
W
\end{bmatrix} =
\begin{bmatrix}
C_\omega & S_\omega & 0 \\
-S_\omega & C_\omega & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & C_I & S_I \\
0 & -S_I & C_I
\end{bmatrix}
\begin{bmatrix}
C_\Omega & S_\Omega & 0 \\
-S_\Omega & C_\Omega & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

(4)

or,

\[
\begin{bmatrix}
P \\
Q \\
W
\end{bmatrix} =
\begin{bmatrix}
C_\omega C_\Omega - S_\omega C_I S_\Omega & -S_\omega S_\Omega + S_\omega C_I C_\Omega & S_\omega S_I \\
-S_\omega C_\Omega - C_\omega C_I S_\Omega & C_\omega S_\Omega + C_\omega C_I C_\Omega & S_\omega S_I \\
S_I S_\Omega & -S_I C_\Omega & C_I
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

(5)
Hence,
\[
\Phi_{PQW}^{ECI} = \begin{bmatrix}
C_\omega C_\Omega - S_\omega C_i S_\Omega & C_\omega S_\Omega + S_\omega C_i C_\Omega & S_\omega S_i \\
-S_\omega C_\Omega - C_\omega C_i S_\Omega & -S_\omega S_\Omega + C_\omega C_i C_\Omega & C_\omega S_i \\
S_i S_\Omega & -S_i C_\Omega & C_i 
\end{bmatrix}.
\]

Then,
\[
\begin{bmatrix}
P \\ Q \\ W
\end{bmatrix} = \Phi_{PQW}^{ECI} \begin{bmatrix}
X \\ Y \\ Z
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
X \\ Y \\ Z
\end{bmatrix} = \left( \Phi_{PQW}^{ECI} \right)^T \begin{bmatrix}
P \\ Q \\ W
\end{bmatrix}, \quad (6)
\]

and
\[
\Phi_{ECI}^{PQW} = \left( \Phi_{PQW}^{ECI} \right)^T.
\]

If we have unit vectors \( \vec{P}, \vec{Q}, \vec{W} \) expressed in the ECI frame (as we obtain using Gibbs Method for example), then
\[
\vec{P} = X_p \hat{i} + Y_p \hat{j} + Z_p \hat{k},
\]
\[
\vec{Q} = X_q \hat{i} + Y_q \hat{j} + Z_q \hat{k},
\]
\[
\vec{W} = X_w \hat{i} + Y_w \hat{j} + Z_w \hat{k},
\]

and
\[
\begin{bmatrix}
\vec{P} \\ \vec{Q} \\ \vec{W}
\end{bmatrix} = \begin{bmatrix}
X_p & Y_p & Z_p \\ X_q & Y_q & Z_q \\ X_w & Y_w & Z_w
\end{bmatrix} \begin{bmatrix}
\hat{i} \\ \hat{j} \\ \hat{k}
\end{bmatrix}. \quad (7)
\]

Where \( \vec{P}, \vec{Q}, \vec{W} \) and \( \hat{i}, \hat{j}, \hat{k} \) are unit vectors along the P, Q, W and X, Y, Z axes respectively and the \( X_p, Y_p, Z_p \) etc. are direction cosines.

Thus, we have \( \Phi_{PQW}^{ECI} \) directly, i.e.,
\[
\Phi_{PQW}^{ECI} = \begin{bmatrix}
X_p & Y_p & Z_p \\ X_q & Y_q & Z_q \\ X_w & Y_w & Z_w
\end{bmatrix}. \quad (8)
\]
Also, note that $\overline{P}$ lies along the eccentricity vector and $\overline{W}$ lies along the angular momentum vector. Consequently, if the position and velocity vectors are available in the ECI frame. We may obtain $\overline{P}, \overline{Q}, \overline{W}$ directly as follows

\[
\overline{e} = \frac{1}{\mu} \left( \mathbf{e} - \frac{\mu \overline{P}}{r} \right), \tag{9}
\]

\[
\overline{P} = \overline{e} / |\overline{e}|, \tag{10}
\]

\[
\overline{h} = \mathbf{h} \times \mathbf{e}, \tag{11}
\]

\[
\overline{W} = \overline{h} / |\overline{h}|, \tag{12}
\]

and

\[
\overline{Q} = \overline{W} \times \overline{P} \quad \text{since} \quad \overline{P}, \overline{Q}, \overline{W} \quad \text{form a right hand triad.} \tag{13}
\]

3. Another frame often used is the RIC frame with unit vectors $\overline{R}, \overline{I}$ and $\overline{C}$. In this frame $\overline{R}$ lies along the instantaneous radius vector, $\overline{I}$ lies in the orbit plane normal to $\overline{R}$ and in the direction of motion of the spacecraft. $\overline{C}$ is normal to the orbit plane and lies along the angular momentum vector. Hence, $\overline{C} = \overline{W}$ of the PQW frame. $\overline{R}$ and $\overline{I}$ align with $\overline{P}$ and $\overline{Q}$ when the spacecraft is at perigee. Although $\overline{R}$ and $\overline{I}$ rotate with the spacecraft radius vector, at each instant in time the frame is considered fixed, so we do not differentiate these unit vectors when transforming velocity to this frame i.e., velocity magnitude in this frame has the same value as in the inertial, ECI, frame. This frame is useful for displaying the difference between two orbits in the radial, in-track, and cross-track directions. To obtain the transformation between the ECI and RIC frames, assume we are given the position and velocity vectors in the ECI frame. Then

\[
\mathbf{r} = X \mathbf{i} + Y \mathbf{j} + Z \mathbf{k}, \tag{14}
\]

\[
\mathbf{r}^\text{eci} = \mathbf{r} = \mathbf{r}^\text{eci}, \tag{15}
\]

\[
\mathbf{h} = \mathbf{r} \times \mathbf{r}^\text{eci}, \tag{16}
\]

and
\[
\mathbf{R} = \frac{\vec{r}}{|\vec{r}|} = \frac{X}{r} \hat{i} + \frac{Y}{r} \hat{j} + \frac{Z}{r} \hat{k}, \quad (17)
\]

\[
\mathbf{C} = \frac{\vec{h}}{|\vec{h}|} = \frac{\mathbf{R} \times \mathbf{E}_{\text{ECI}}}{|\mathbf{R} \times \mathbf{E}_{\text{ECI}}|}, \quad (18)
\]

\[I = \mathbf{C} \times \mathbf{R}. \quad (19)\]

Or in matrix notation
\[
\begin{bmatrix}
\mathbf{R} \\
\mathbf{I} \\
\mathbf{C}
\end{bmatrix} = 
\begin{bmatrix}
R_x & R_y & R_z \\
I_x & I_y & I_z \\
C_x & C_y & C_z
\end{bmatrix}
\begin{bmatrix}
\vec{i} \\
\vec{j} \\
\vec{k}
\end{bmatrix}, \quad (20)
\]

Define
\[
\mathbf{\beta}_{\text{RIC}}^{\text{ECI}} = 
\begin{bmatrix}
R_x & R_y & R_z \\
I_x & I_y & I_z \\
C_x & C_y & C_z
\end{bmatrix},
\]

where the elements of \( \mathbf{\beta}_{\text{RIC}}^{\text{ECI}} \) are the direction cosines between the ECI and RIC frames and are given by Eqns (17), (18), and (19).

Hence, the coordinates in one frame can be computed from coordinate in the other, i.e.,
\[
\begin{bmatrix}
\mathbf{R} \\
\mathbf{I} \\
\mathbf{C}
\end{bmatrix} = \mathbf{\beta}_{\text{RIC}}^{\text{ECI}}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = \mathbf{\beta}_{\text{RIC}}^{\text{ECI}}^T
\begin{bmatrix}
\mathbf{R} \\
\mathbf{I} \\
\mathbf{C}
\end{bmatrix}. \quad (21)
\]

If we wish to examine the difference between two orbits in the \( \mathbf{R}, \mathbf{I}, \mathbf{C} \) directions, we do the following. Designate one orbit as the reference orbit. Call the position and velocity in this orbit \( \vec{r}^* \) and \( \vec{\dot{r}}^* \). Then use equation (17), (18), (19) and (20) to compute the unit vectors \( \mathbf{R}, \mathbf{I}, \mathbf{C} \) and \( \mathbf{\beta}_{\text{RIC}}^{\text{ECI}} \) using \( \vec{r}^* \) and \( \vec{\dot{r}}^* \). Next compute position and velocity difference in the ECI frame,
\[
\begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix} = \left[ \Delta \vec{r} \right]_{\text{ECI}} = \left[ \vec{r}^* - \vec{\dot{r}}^* \right]_{\text{ECI}}, \quad
\begin{bmatrix}
\Delta \dot{X} \\
\Delta \dot{Y} \\
\Delta \dot{Z}
\end{bmatrix} = \left[ \Delta \dot{\vec{r}} \right]_{\text{ECI}} = \left[ \vec{\dot{r}}^* - \vec{\dot{\dot{r}}} \right]_{\text{ECI}}. \quad (22)
\]

Then use equation (21) to compute differences in the radial, in-track and cross-track directions in position and velocity i.e.,
\[
\begin{bmatrix}
\Delta R \\
\Delta I \\
\Delta C
\end{bmatrix} = \beta_{R_{IC}}^{ECI} \begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix}, \quad \begin{bmatrix}
\Delta R \\
\Delta I \\
\Delta C
\end{bmatrix} = \beta_{R_{IC}}^{ECI} \begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix}.
\] (23)