Web Page: http://www.colorado.edu/physics/phys7840

NOTE: Next lectures Tuesday, Sept. 26; noon
Thursday, Sept. 28; noon
Tuesday, Oct. 3; noon
Thursday, Oct. 5; noon
more????

Today’s topics: Bianchi Identities
Covariant divergence of Ricci tensor
Einstein field equations
linearized equations
first-order coordinate transformations
free space equations
gravity waves
detection of gravity waves with clocks??
Ricci tensor

In General Relativity, the components of the metric tensor (10 independent quantities) are treated as potentials, and one wants to derive field equations that determine these 10 quantities. The Ricci tensor is a candidate, since it is symmetric with respect to exchange of the two indices. There is only one independent way to get a second-rank tensor from the curvature tensor by contraction:

$$R_{\sigma\rho} = R^\alpha_{\sigma\alpha\rho} = R_{\rho\sigma} \quad \text{(Ricci Tensor)}$$
One last set of identities is very important in GR. The Bianchi identities state:

\[ R^\mu_{\alpha\beta\gamma;\delta} + R^\mu_{\alpha\gamma\delta;\beta} + R^\mu_{\alpha\delta\beta;\gamma} = 0. \]

This is the sum of the covariant derivatives of the curvature tensor with last three indices cyclically permuted. To prove this, write out the sum at a point where the Christoffel symbols of the second kind have been reduced to zero by a coordinate transformation:

\[ R^\mu_{\alpha\beta\gamma;\delta} = \frac{\partial^2 \Gamma^\mu_{\alpha\gamma}}{\partial x^\delta \partial x^\beta} - \frac{\partial^2 \Gamma^\mu_{\alpha\beta}}{\partial x^\delta \partial x^\gamma}; \quad \beta \rightarrow \gamma \rightarrow \delta \rightarrow \beta \]

\[ R^\mu_{\alpha\gamma\delta;\beta} = \frac{\partial^2 \Gamma^\mu_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} - \frac{\partial^2 \Gamma^\mu_{\alpha\gamma}}{\partial x^\delta \partial x^\beta}; \]

\[ R^\mu_{\alpha\delta\beta;\gamma} = \frac{\partial^2 \Gamma^\mu_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} - \frac{\partial^2 \Gamma^\mu_{\alpha\delta}}{\partial x^\gamma \partial x^\beta}; \]

Adding these three equations proves the equation at the top of the page.
Scalar curvature

One can contract the Ricci tensor further to obtain the “scalar curvature”

\[ R \equiv R^\alpha_\alpha = g^{\alpha\beta} R_{\alpha\beta}. \]
Example: Surface of a sphere

The metric is two-dimensional:

\[ ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2. \]

\begin{align*}
\text{Christoffel}[1,2,2] &= -\text{Cos[theta]} \times \text{Sin[theta]} \\
\text{Christoffel}[2,1,2] &= \text{Cot[theta]} \\
\text{Christoffel}[2,2,1] &= \text{Cot[theta]} \\
\text{Riemann}[1,2,1,2] &= \text{Sin[theta]}^2 \\
\text{Riemann}[1,2,2,1] &= -\text{Sin[theta]}^2 \\
\text{Riemann}[2,1,1,2] &= -1 \\
\text{Riemann}[2,1,2,1] &= 1 \\
\text{Ricci}[1,1] &= 1 \\
\text{Ricci}[2,2] &= \text{Sin[theta]}^2 \\
\text{RicciScalar} &= \frac{2}{a^2}
\end{align*}
Parallel transport on the surface of a sphere

Suppose we take a parallel of latitude at angle $\theta$, and transport a vector parallel to itself around through some angle. The differential equation of parallel transport of the vector is

$$dt^\mu = \delta_{\parallel} t^\mu = -\Gamma^\mu_{\alpha\sigma} t^\alpha dx^\sigma;$$

$$\frac{dt^\mu}{d\lambda} + \Gamma^\mu_{\alpha\sigma} t^\alpha \frac{dx^\sigma}{d\lambda} = 0.$$ 

We consider parallel propagation keeping $\theta$ fixed.

$$\frac{dt^\mu}{d\lambda} + \Gamma^\mu_{\alpha\sigma} t^\alpha \frac{d\phi}{d\lambda} = 0;$$

$$\frac{dt^\mu}{d\phi} + \Gamma^\mu_{\alpha\sigma} t^\alpha = 0.$$ 

We get two coupled equations for the components of the vector:

$$\frac{dt^1}{d\phi} - \cos \theta \sin \theta t^2 = 0;$$

$$\frac{dt^2}{d\phi} + \frac{\cos \theta}{\sin \theta} t^1 = 0;$$
Parallel transport on the surface of a sphere

The solutions to the differential equations, putting in boundary conditions such that there is only a 2-component at the start where the azimuthal angle is zero, and also that the vector is normalized, are:

\[ t^1 = \frac{1}{a} \sin(\cos \theta \phi), \quad t^2 = \frac{1}{a \sin \theta} \cos(\cos \theta \phi). \]

As the vector is transported along the parallel, which is not a geodesic, it rotates.
Parallel transport along an arbitrary path

The vector rotates in a retrograde direction; at each point along the path there is a geodesic that is tangent to the curve. During each infinitesimal path increment the tangent vector maintains a constant angle with respect to the local geodesic (great circle).
Example: Divergence of a gradient

The tensor analysis developed so far allows one to construct physically interesting quantities such as the generalization of the Laplacian. This is a second-order differential operator that operates on a scalar. First we take the gradient of a scalar:

$$\frac{\partial \Phi}{\partial x^\mu} = \Phi_{,\mu} = \Phi_{;\mu};$$

We want to take the divergence of this quantity but first the index has to be raised to get a contravariant vector:

$$g^{\kappa\mu} \Phi_{,\mu}$$

The covariant derivative of this is

$$\left( g^{\kappa\mu} \Phi_{,\mu} \right)_{;\lambda}$$

and to get a second-order scalar differential operator, we set $\lambda = \kappa$ and sum.
Divergence of a gradient

\[
\left( g^{\kappa\mu} \Phi_{,\mu} \right)_{;\kappa} = \partial_{\kappa} \left( g^{\kappa\mu} \Phi_{,\mu} \right) + \Gamma_{\beta\kappa}^{\kappa} g^{\beta\mu} \Phi_{,\mu}.
\]

The contracted quantity \( \Gamma_{\beta\kappa}^{\kappa} \) occurs often and we shall need an expression for it.

\[
\Gamma_{\beta\kappa}^{\kappa} = \frac{1}{2} g^{\kappa\lambda} \left( g_{\lambda\beta,\kappa} + g_{\lambda,\beta\kappa} - g_{\beta\kappa,\lambda} \right)
\]

\[
\Gamma_{\beta\kappa}^{\kappa} = \frac{1}{2} g^{\kappa\lambda} g_{\lambda,\beta\kappa} = \frac{1}{2} g^{\kappa\lambda} \frac{\partial}{\partial x^\beta} g_{\lambda\kappa}
\]

\[
= \frac{1}{2} \frac{1}{\det(g_{\alpha\beta})} \sum_{\lambda,\kappa} \text{(Cofactor } g_{\lambda\kappa}) \frac{\partial}{\partial x^\beta} g_{\lambda\kappa}
\]

\[
= \frac{1}{2} \frac{1}{\det(g_{,\beta})} \frac{\partial}{\partial x^\beta} \det(g_{,\beta}) = \frac{\partial}{\partial x^\beta} \ln \sqrt{-\det(g_{,\beta})} = \frac{\partial}{\partial x^\beta} \ln \sqrt{-g}
\]
Thus the scalar divergence of a gradient in generalized coordinates is
\[
\left(g^{\kappa \mu} \Phi_{.\mu}\right)_{,\kappa} = \partial_{\kappa} \left(g^{\kappa \mu} \Phi_{.\mu}\right) + \Gamma_{\beta \kappa}^{\kappa} g^{\beta \mu} \Phi_{.\mu}
\]
\[
= \partial_{\kappa} \left(g^{\kappa \mu} \Phi_{.\mu}\right) + \frac{1}{\sqrt{-g}} \partial_{\beta} \left(\sqrt{-g} g^{\beta \mu} \Phi_{.\mu}\right)
\]
\[
= \frac{1}{\sqrt{-g}} \partial_{\beta} \left(\sqrt{-g} g^{\beta \mu} \Phi_{.\mu}\right).
\]
This works for any type of orthogonal coordinate system in three dimensions, e.g., spherical polar coordinates:
\[
ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2;
\]
\[
g_{\mu \nu} = \begin{pmatrix}
1 & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}; \quad \sqrt{+g} = r^2 \sin \theta.
Laplacian in spherical polar coordinates

\[
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}; \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}; \quad \sqrt{+g} = r^2 \sin \theta
\]

\[
\frac{1}{\sqrt{g}} \partial_{\kappa} \left( \sqrt{g} g^{\kappa\mu} \Phi_{,\mu} \right) = \frac{1}{r^2 \sin \theta} \left( \partial_r \left( r^2 \sin \theta \partial_r \Phi \right) + \partial_{\theta} \left( \frac{r^2 \sin \theta}{r^2} \partial_{\theta} \Phi \right) + \partial_{\varphi} \left( \frac{r^2 \sin \theta}{r^2 \sin^2 \theta} \partial_{\varphi} \Phi \right) \right)
\]
Einstein Field Equations

In Newtonian mechanics and electromagnetic theory, conservation of energy and momentum is expressed by four equations arising from the symmetric stress-energy tensor:

\[ T^{\mu\nu} = 0. \]

In generalized coordinates in the presence of gravitation, this ought to become

\[ T^{\mu\nu}_{\ ;\nu} = 0. \]

If we try field equations of the form

\[ R^{\mu\nu} = \kappa T^{\mu\nu}, \]

Then we run into trouble with the Bianchi identities since

\[ R^{\mu\nu}_{\ ;\nu} \neq 0. \]
Contracting Bianchi identities

Let’s compute the covariant divergence of the Ricci tensor by contracting the Bianchi identities:

\[ R_{\alpha\beta\gamma;\delta}^\mu + R_{\alpha\gamma\delta;\beta}^\mu + R_{\alpha\delta\beta;\gamma}^\mu = 0. \]

Set \( \beta = \mu \) and sum, while also raising the index \( \alpha \):

\[ R_{\mu\gamma;\delta}^{\mu\alpha} + R_{\gamma\delta;\mu}^{\mu\alpha} + R_{\delta\mu;\gamma}^{\mu\alpha} = 0. \]

(We made use of the fact that the covariant derivative of the metric tensor vanishes.)

\[ R_{\mu\gamma;\delta}^{\mu\alpha} + R_{\gamma\delta;\mu}^{\mu\alpha} + R_{\delta\mu;\gamma}^{\mu\alpha} = 0. \]
\[ R_{\gamma;\delta}^{\alpha} + R_{\gamma\delta;\mu}^{\mu\alpha} - R_{\mu\delta;\gamma}^{\mu\alpha} = 0. \]
\[ R_{\gamma;\delta}^{\alpha} + R_{\gamma\delta;\mu}^{\mu\alpha} - R_{\delta;\gamma}^{\alpha} = 0. \]
Contracting the Bianchi identities

\[ R^\alpha_{\gamma;\delta} + R^{\mu\alpha}_{\gamma;\mu} - R^\alpha_{\delta;\gamma} = 0. \]

\[ R^\alpha_{\gamma;\delta} + (-1)^2 R^{\alpha\mu}_{\delta\gamma;\mu} - R^\alpha_{\delta;\gamma} = 0. \]

Now set \( \alpha = \delta \) and contract:

\[ R^\alpha_{\gamma;\alpha} + R^{\mu}_{\gamma;\mu} - R^{\mu}_{\gamma;\mu} = 2R^{\mu}_{\gamma;\mu} - R^{\mu}_{\gamma;\mu} = 0. \]

Thus we conclude that

\[ R^{\mu}_{\gamma;\mu} - \frac{1}{2} R^{\mu}_{\gamma;\mu} = 0. \]

This can also be written as

\[ R^{\mu}_{\gamma;\mu} - \frac{1}{2} \delta^\mu_{\gamma} R^{\mu}_{;\mu} = 0. \]
We define the Einstein Tensor as

\[ G^\mu_\gamma = R^\mu_\gamma - \frac{1}{2} \delta^\mu_\gamma R. \]

This has vanishing covariant divergence. The fully covariant form of the Einstein tensor is obtained by lowering the contravariant index:

\[ G_\beta\gamma = R_\beta\gamma - \frac{1}{2} g_\beta\gamma R. \]

This apparently has 10 symmetric components, and second derivatives of the metric tensor enter. It’s reasonable to think that the equations determining the metric tensor could be of the form

\[ G_\beta\gamma = \kappa T_\beta\gamma, \]

where \( \kappa \) is a coupling constant. Then conservation laws could be expressed by

\[ G^\mu_\gamma;\mu = \kappa T^\mu_\gamma;\mu = 0, \]
Einstein Field Equations

Einstein pointed out that it is possible another term could enter the field equations, that would also vanish when taking the covariant divergence. The field equations would be

\[ G_{\beta\gamma} + \Lambda g_{\beta\gamma} = \kappa T_{\beta\gamma}, \]

where \( \Lambda \) is the “cosmological constant.” We’ll ignore this term even though it is important when discussing dark matter. Also, we’ll put off dealing with the actual form of the stress-energy tensor of matter until later, and look for solutions of the field equations in empty space. So for example, for gravitational waves or outside a star, we’re interested in

\[ G_{\beta\gamma} = 0. \]
Does it work???

**Linearized coordinate transformations**

Anticipating needed coordinate transformation equations that will arise a little later, suppose we approximate the metric tensor by small deviations from the Minkowski metric:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \]

where \( h_{\mu\nu} \) is not a tensor, but is small enough that terms of second order can be neglected. Let us make a correspondingly small change in coordinates, of the form

\[ x'^{\mu} = x^{\mu} - \xi^{\mu} (x'); \]

Then

\[ \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} - \xi^{\mu}_{,\nu}. \]

We’ll substitute these expressions in the transformation law for the metric tensor and neglect higher-order terms.
Linearized transformation of metric tensor

\[ g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} \left( \eta_{\alpha\beta} + h_{\alpha\beta} \right) \]

\[ = \left( \delta^\alpha_{\mu} - \xi^\alpha_{\mu,\nu} \right) \left( \delta^\beta_{\nu} - \xi^\beta_{\nu,\mu} \right) \left( \eta_{\alpha\beta} + h_{\alpha\beta} \right) \]

\[ \approx \eta_{\mu\nu} + h_{\mu\nu} - \delta^\alpha_{\mu} \xi^\beta_{\nu,\nu} \eta_{\alpha\beta} - \delta^\beta_{\nu} \xi^\alpha_{\mu,\nu} \eta_{\alpha\beta} \]

\[ = \eta_{\mu\nu} + h_{\mu\nu} - \eta_{\mu\beta} \xi^\beta_{\nu,\nu} - \eta_{\alpha\nu} \xi^\alpha_{\mu,\nu} \cdot \]

\[ h'_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\beta} \xi^\beta_{\nu,\nu} - \eta_{\alpha\nu} \xi^\alpha_{\mu,\nu} \cdot \]

While we’re at it, we raise the index \( \mu \) and contract, defining

\[ h' = \eta^{\nu\mu} h'_{\mu\nu} ; \quad h = \eta^{\nu\mu} h_{\mu\nu} . \]

Then

\[ h' = h - \delta^\mu_{\beta} \xi^\beta_{\mu,\nu} - \delta^\mu_{\alpha} \xi^\alpha_{\mu,\nu} = h - 2 \xi^\mu_{,\mu} . \]
Getting ready to linearize

Anticipating what is going to arise in linearizing the Einstein tensor, we define

$$\sigma_{\mu} = \eta^{\gamma\delta} \left( h_{\mu\gamma,\delta} - \frac{1}{2} \eta_{\gamma\delta} h_{,\delta} \right) = \eta^{\gamma\delta} h_{\mu\gamma,\delta} - \frac{1}{2} h_{,\mu}. $$

Making the first-order coordinate transformation,

$$\sigma'_{\mu} = \eta^{\gamma\delta} h'_{\mu\gamma,\delta} - \frac{1}{2} h',_{\mu}$$

$$= \eta^{\gamma\delta} \left( h_{\mu\gamma,\delta} - \eta_{\mu\alpha} \xi_{\alpha}^{\gamma} - \eta_{\gamma\alpha} \xi_{\alpha}^{\delta} \right) - \frac{1}{2} \left( h_{,\mu} - 2 \xi_{\alpha}^{\alpha},_{\alpha\mu} \right)$$

$$= \eta^{\gamma\delta} h_{\mu\gamma,\delta} - \frac{1}{2} h_{,\mu} - \eta^{\gamma\delta} \eta_{\mu\alpha} \xi_{\alpha}^{\gamma} - \eta^{\gamma\delta} \eta_{\gamma\alpha} \xi_{\alpha}^{\delta} + \xi_{\alpha}^{\alpha},_{\alpha\mu}$$

But the last two terms cancel:

$$-\eta^{\gamma\delta} \eta_{\gamma\alpha} \xi_{\alpha}^{\delta},_{\mu\delta} + \xi_{\alpha}^{\alpha},_{\alpha\mu} = -\delta_{\alpha}^{\delta} \xi_{\alpha}^{\alpha},_{\mu\delta} + \xi_{\alpha}^{\alpha},_{\alpha\mu} = -\xi_{\alpha}^{\alpha},_{\mu\alpha} + \xi_{\alpha}^{\alpha},_{\alpha\mu} = 0.$$
New coordinate conditions

\[ \sigma'_{\mu} = \eta^{\gamma\delta} h_{\mu\gamma,\delta} - \frac{1}{2} h_{,\mu} - \eta^{\gamma\delta} \eta_{\mu\alpha} \xi^\alpha,_{\gamma\delta} \]

\[ = \sigma_{\mu} - \eta^{\gamma\delta} \left( \eta_{\mu\alpha} \xi^\alpha \right),_{\gamma\delta} \]

But the last term is just the D'Alembertian operator applied to \( \xi_{\mu} \equiv \left( \eta_{\mu\alpha} \xi^\alpha \right) \).

\[ \eta^{\gamma\delta} \left( \xi_{\mu} \right),_{\gamma\delta} = \left( -\frac{\partial^2}{c^2 \partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \xi_{\mu} \]

We can therefore choose a coordinate transformation such that

\[ \eta^{\gamma\delta} \left( \eta_{\mu\alpha} \xi^\alpha \right),_{\gamma\delta} = \sigma_{\mu} \Rightarrow \sigma'_{\mu} = 0 \]

since solutions of the wave equation with sources has a well-known solution that can be obtained by integration (e.g., see Electromagnetic Theory).
Gauge freedom

\[ \sigma_\mu = \eta^{\gamma\delta} \left( \xi_\mu \right), \gamma^\delta = \Box^2 \xi_\mu \]

We can choose \( \xi_\mu \) so that this equation is satisfied. Further, we can add an arbitrary solution of the wave equation,

\[ \Box^2 \xi_\mu = 0 \]

to \( \xi_\mu \) without changing the value of \( \sigma'_\mu \).
Linearized Field equations

So whenever we see a quantity $\sigma_\mu$ arising in linearizing the field equations, we can say that it can be eliminated by an appropriately chosen coordinate transformation.

Now we are finally ready to linearize. We write

$$g_{\gamma\delta} = \eta_{\gamma\delta} + h_{\gamma\delta},$$

where $h$ is small, and all the calculations are carried only to first order of smallness. The Christoffel symbols involve derivatives of the metric tensor and so will be of first order. For example,

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} \left( g_{\sigma\alpha, \beta} + g_{\sigma\beta, \alpha} - g_{\alpha\beta, \sigma} \right) \approx \frac{1}{2} \eta^{\mu\sigma} \left( h_{\sigma\alpha, \beta} + h_{\sigma\beta, \alpha} - h_{\alpha\beta, \sigma} \right).$$
Does it work??  Linearize

Then the non-linear terms in the Riemann tensor will be negligible:

\[
R^\mu_{\alpha\rho\sigma} = \partial_\rho \Gamma^\mu_{\alpha\sigma} - \partial_\sigma \Gamma^\mu_{\alpha\rho} + \Gamma^\mu_{\beta\rho} \Gamma^\beta_{\alpha\sigma} - \Gamma^\mu_{\beta\sigma} \Gamma^\beta_{\alpha\rho} \\
\approx \partial_\rho \Gamma^\mu_{\alpha\sigma} - \partial_\sigma \Gamma^\mu_{\alpha\rho} \\
= \frac{1}{2} \eta^{\mu\kappa} \left( h_{\kappa\alpha,\rho} + h_{\kappa\sigma,\alpha} - h_{\alpha\sigma,\kappa} \right) - \frac{1}{2} \eta^{\mu\kappa} \left( h_{\kappa\alpha,\rho\sigma} + h_{\kappa\rho,\alpha\sigma} - h_{\alpha\rho,\kappa\sigma} \right); \\
R^\mu_{\alpha\rho\sigma} = \frac{1}{2} \eta^{\mu\kappa} \left( h_{\kappa\sigma,\alpha\rho} - h_{\alpha\sigma,\kappa\rho} - h_{\kappa\rho,\alpha\sigma} + h_{\alpha\rho,\kappa\sigma} \right).
\]
Linearized Field Equations

\[ R^\mu_{\alpha\rho\sigma} = \frac{1}{2} \eta^{\mu\kappa} \left( h_{\kappa\sigma,\alpha\rho} - h_{\alpha\sigma,\kappa\rho} - h_{\kappa\rho,\alpha\sigma} + h_{\alpha\rho,\kappa\sigma} \right). \]

Construct the Ricci tensor by contraction: \( \rho = \mu. \)

\[ R_{\alpha\sigma} = \frac{1}{2} \eta^{\mu\kappa} \left( h_{\kappa\sigma,\alpha\mu} - h_{\alpha\sigma,\kappa\mu} - h_{\kappa\mu,\alpha\sigma} + h_{\alpha\mu,\kappa\sigma} \right) \]

And the scalar curvature by raising the index \( \alpha \) and contracting:

\[ R = \eta^{\alpha\sigma} R_{\alpha\sigma} = \frac{1}{2} \eta^{\alpha\sigma} \eta^{\mu\kappa} \left( h_{\kappa\sigma,\alpha\mu} - h_{\alpha\sigma,\kappa\mu} - h_{\kappa\mu,\alpha\sigma} + h_{\alpha\mu,\kappa\sigma} \right) \]

\[ = \frac{1}{2} \eta^{\alpha\sigma} \eta^{\mu\kappa} \left( h_{\kappa\sigma,\alpha\mu} - h_{\alpha\sigma,\kappa\mu} - h_{\kappa\mu,\alpha\sigma} + h_{\alpha\mu,\kappa\sigma} \right) \]

\[ = h^{\mu\alpha}_{\ ,\alpha\mu} - \Box^2 h \]
Construct the linearized Einstein tensor

\[
G_{\alpha\sigma} = R_{\alpha\sigma} - \frac{1}{2} g_{\alpha\sigma} R
\]

\[
= \frac{1}{2} \eta^{\mu\kappa} (h_{\kappa\sigma,\alpha\mu} - h_{\alpha\sigma,\kappa\mu} - h_{\kappa\mu,\alpha\sigma} + h_{\alpha\mu,\kappa\sigma}) - \frac{1}{2} \eta_{\alpha\sigma} (h_{\mu\beta},_{\beta\mu} - \Box^2 h)
\]

Recall that

\[
\sigma_\mu = \eta^{\kappa\delta} \left( h_{\mu\gamma,\delta} - \frac{1}{2} \eta_{\mu\gamma} h_{,\delta} \right) = \eta^{\kappa\delta} h_{\mu\gamma,\delta} - \frac{1}{2} h_{,\mu}.
\]

So

\[
\eta^{\kappa\delta} h_{\mu\gamma,\delta} = \sigma_\mu + \frac{1}{2} h_{,\mu}; \quad \eta^{\kappa\delta} h_{\mu\gamma,\delta\sigma} = \sigma_{\mu,\sigma} + \frac{1}{2} h_{,\mu\sigma}
\]

We’ll rewrite each of the six terms in the linearized Einstein tensor.
Rewrite of linearized Einstein tensor

\[ \eta^{\gamma\delta} h_{\mu\gamma,\delta\sigma} = \sigma_{\mu,\sigma} + \frac{1}{2} h_{\mu\sigma}. \]

\[ G_{\alpha\sigma} = \frac{1}{2} \eta^{\mu\kappa} \left( h_{\kappa\sigma,\alpha\mu} - h_{\alpha\sigma,\kappa\mu} - h_{\kappa\mu,\alpha\sigma} + h_{\alpha\mu,\kappa\sigma} \right) - \frac{1}{2} \eta_{\alpha\sigma} \left( h^{\mu\beta},_{\beta\mu} - \Box^2 h \right) \]

\[ \frac{1}{2} \eta^{\mu\kappa} h_{\kappa\sigma,\alpha\mu} = \frac{1}{2} \left( \sigma_{\sigma,\alpha} + \frac{1}{2} h_{\sigma\alpha} \right); \quad -\frac{1}{2} \eta^{\mu\kappa} h_{\alpha\sigma,\kappa\mu} = -\frac{1}{2} \Box^2 h_{\alpha\sigma} \]

\[ -\frac{1}{2} \eta^{\mu\kappa} h_{\kappa\mu,\alpha\sigma} = -\frac{1}{2} h_{\alpha\sigma} \quad \frac{1}{2} \eta^{\mu\kappa} h_{\alpha\mu,\kappa\sigma} = \frac{1}{2} \left( \sigma_{\alpha,\sigma} + \frac{1}{2} h_{\alpha\sigma} \right) \]

\[ -\frac{1}{2} \eta_{\alpha\sigma} \left( h^{\mu\beta},_{\beta\mu} \right) = -\frac{1}{2} \eta_{\alpha\sigma} \left( \eta^{\mu\gamma} \eta^{\beta\delta} h_{\gamma\delta,\beta\mu} \right) = -\frac{1}{2} \eta_{\alpha\sigma} \eta^{\mu\nu} \left( \sigma_{\gamma,\mu} + \frac{1}{2} h_{\gamma\mu} \right) = -\frac{1}{2} \eta_{\alpha\sigma} \eta^{\mu\nu} \sigma_{\gamma,\mu} - \frac{1}{4} \eta_{\alpha\sigma} \Box^2 h \]

\[ -\frac{1}{2} \eta_{\alpha\sigma} (\Box^2 h) = +\frac{1}{2} \eta_{\alpha\sigma} (\Box^2 h) \]
Linearized Einstein tensor at last!

Adding all the contributions:

\[
G_{\alpha\sigma} = \frac{1}{2} \left( \sigma_{\sigma,\alpha} + \frac{1}{2} h_{,\sigma\alpha} \right) - \frac{1}{2} \Box^2 h_{\alpha\sigma} - \frac{1}{2} h_{,\alpha\sigma} \\
\frac{1}{2} \left( \sigma_{\alpha,\sigma} + \frac{1}{2} h_{,\alpha\sigma} \right) - \frac{1}{2} \eta_{\alpha\sigma} \eta_{,\gamma,\mu} \sigma_{\gamma,\mu} - \frac{1}{4} \eta_{\alpha\sigma} \Box^2 h + \frac{1}{2} \eta_{\alpha\sigma} \left( \Box^2 h \right)
\]

\[
G_{\alpha\sigma} = -\frac{1}{2} \Box^2 \left( h_{\alpha\sigma} - \frac{1}{2} \eta_{\alpha\sigma} h \right) = \kappa T_{\alpha\sigma}
\]

These are the linearized field equations. There is also the coordinate condition

\[
\eta^{\gamma\delta} h_{\mu\gamma,\delta} - \frac{1}{2} h_{,\mu} = 0.
\]