Ten Lectures on Relativity

Neil Ashby
NIST Affiliate
(303) 497-4395
ashby@boulder.nist.gov

WEB PAGE FOR THESE LECTURES:

http://www.colorado.edu/physics/phys7840
Outline of lecture 1

Fundamental principles 3-4
Simultaneity & Longitudinal Doppler 5-10
Gravitational frequency shifts 11-15
Time dilation--16-18
Notation; summation convention 19
Minkowski metric 20
Proper time; variational principle 21-22
Lorentz transformations 23
Velocity transformations 24
Fundamental Principles

• **Principle of inertia**
  – All inertial frames are equivalent regarding the expression of all laws of physics

• **Constancy of the speed of light**
  – The speed of light, c, is a constant independent of the motion of the source (or of the observer)

• **Principle of Equivalence (“weak form”)**
  – Over a small region of space and time, the fictitious gravitational field induced by acceleration cannot be distinguished from a real gravitational field due to a mass.
Clock and Rod Hypotheses

Clock Hypothesis:

Over a small region of space and time, measurements made with an accelerated clock give results identical to measurements made with an instantaneously comoving clock that is not accelerated.

Rod Hypothesis:

Over a small region of space and time, measurements made with an accelerated rod give results identical to measurements made with an instantaneously comoving rod that is not accelerated.
**Constancy of \( c \)—source of breakdown of absolute simultaneity**

- Event at \( t=t'=0 \)

Signal propagating with speed \( c \)
- Catches up to end with relative speed \( c - v \)

Time at which signal catches up to end of rod is

\[
t = \frac{L}{c - v} = \frac{L}{c} \left(1 - \frac{v}{c}\right)^{-1} \approx \frac{L}{c} + \frac{vL}{c^2} = t' + \frac{vL}{c^2}.
\]

\[
t' = t - \frac{vL}{c^2}.
\]
Relativity of Simultaneity

To an observer on the ground, let two lightning strokes at the front and back of the train be simultaneous.

The “moving” observer at the train’s midpoint finds the event at front occurs first.

\[ t' \approx t - \frac{vx}{c^2} \]
Breakdown of simultaneity

\[ t' \simeq t - \frac{vx}{c^2} \]

Standard configuration of relatively moving axes
Breakdown of simultaneity implies Doppler effect

For two events that are simultaneous in $S$, $S'$ thinks the one to the right is too early, so must allow additional time before marking the wave fronts. The additional time is

$$\Delta t' = \frac{vx}{c^2} = \frac{v\lambda}{c^2} = \frac{\lambda' - \lambda}{c};$$

$$\lambda' = \lambda(1 + \frac{v}{c}).$$

(leads to Doppler frequency shift)
Doppler effect implies breakdown of simultaneity

If time were absolute and the observers agreed about simultaneity, there would be no wavelength change. Assume the wavelength difference is

\[ \lambda' - \lambda = \frac{v\lambda}{c}. \]

Then S says that S’ marked wavefront #1 too late, by an amount

\[ \Delta t = \frac{\lambda' - \lambda}{c} = \frac{v\lambda}{c} = \frac{vx}{c^2} \]
Doppler frequency shift

\[ c = f \lambda, \quad \ln c = \ln f + \ln \lambda. \]

So when lambda increases, the frequency decreases. Taking the differential of the logarithm function gives

\[ \frac{\Delta f}{f} = - \frac{\Delta \lambda}{\lambda} = - \frac{v}{c}. \]
Over a small region of space and time, a fictitious gravity field induced by acceleration cannot be distinguished from a gravity field produced by mass.
Gravitational Frequency Shift
Gravitational Frequency Shift

The situation in the rocket is static. The fractional frequency difference between the clocks is

\[ \frac{\Delta f}{f} = + \frac{\Delta \Phi}{c^2} . \]
Frequency shifts due to Gravitational Potential Differences

Let $\Phi_0$ be the gravitational potential on earth’s geoid--at mean sea level, and $r$ be the radius of a GPS satellite.

$$\Delta \Phi = -\frac{GM_E}{r} - \Phi_0$$

$\Phi_0$ includes contributions from earth’s oblateness as well as its mass:

$$\Phi_0 = -\frac{GM}{a_1}(1 + \frac{1}{2}J_2),$$

where: $a_1$ is the equatorial radius of the earth;

$J_2 \approx .001086$ is earth’s quadrupole moment coefficient.
Gravitational frequency shifts in the GPS

To get a rough estimate, assume the satellite orbit is circular and the reference clock is on earth’s equator.

\[
\frac{\Delta f}{f} = \frac{1}{c^2} \left( -\frac{GM}{a} - \left( -\frac{GM}{a_1} \right) \right)
\]

where \( GM = 3.986 \times 10^{14} \text{ m}^3/\text{s}^2; \)

\( a = 26,562 \text{ km}; \)

\( a_1 = 6,378 \text{ km} \)

\[
\frac{\Delta f}{f} \approx 5 \times 10^{-10}; \quad (\approx 13 \text{ km navigation error per day})
\]
The constancy of the speed of light implies time dilation
Pythagorean Theorem + Constancy of $c$

We can let $L$ approach 0.
Equivalence of Lorentz Contraction & Time Dilation

Unstable particles travel down a tube and get through before they decay.

\[ \tau_0 = \text{Lifetime of unstable particle when at rest.} \]

\[ V \tau < L_0 \]

\[ V \tau = \frac{V \tau_0}{\sqrt{1 - \frac{v^2}{c^2}}} > L_0 \]

To an observer traveling with the particles, the tube appears shortened:

\[ L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \]

\[ V \tau_0 > L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \]
Greek indices: $\mu, \nu, \alpha, \beta = 0$ to 3;

Latin indices: $I, j, k, l$—1 to 3;

0 index: coordinate time, $x^0 = ct$; $x^k$ for $k=1,2,3$ represents $\{x,y,z\}$;

Einstein summation convention:

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu = A_0 B^0 + A_k B^k$$

$$= A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3.$$  

$A_\mu B^\mu$ No!

Small spherical light wave starting from origins:

$$-(c\, dt')^2 + (dx')^2 + (dy')^2 + (dz')^2 = -(c\, dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$
Minkowski Metric

\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta'_{\mu\nu} \]

Small spherical light wave starting from origins:

\[ \eta_{\mu\nu} dx^\mu \, dx^\nu = \eta_{\mu\nu} dx^\mu \, dx^\nu = -ds^2 \]

The negative sign is chosen to correspond to the negative sign of The 00-component of the Minkowski metric.

\( ds^2 \) (or \( ds \)) is the fundamental scalar of special relativity.
Proper Time

Consider a particle with a clock attached, moving with speed $v$ relative to the “lab.”

$$\Delta \tau = \frac{1}{c} \int_{P_1}^{P_2} ds;$$

Along the actual path of a free particle, the proper time will be a minimum; any other path will take longer. So there is
Variational Principle

Along the actual path of a free particle, the proper time will be a minimum; any other path between the same two endpoints in spacetime will involve a higher velocity, so the proper time elapsed will be less. So there is a variational principle to describe the actual path:

\[ \delta \int_{P_1}^{P_2} ds = 0. \]
Lorentz Transformations

Relativity of Simultaneity + Time Dilation + Lorentz Contraction + Constancy of $c$ give rise to the “Lorentz Transformations:”

\[
ct' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (ct - \frac{v}{c} x);
\]

\[
x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - \frac{v}{c} ct);
\]

\[y' = y; \quad z' = z.\]

OR

\[x'^\mu = \Lambda^\mu_{\nu} x^\nu, \quad \text{(contravariant components)}\]

\[
\Lambda^\mu_{\nu} = \begin{pmatrix}
\gamma & -\frac{\gamma v}{c} & 0 & 0 \\
-\frac{\gamma v}{c} & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}; \quad \gamma = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}.\]
Velocity Transformations

Use Lorentz Transformations; consider a particle with velocity components \(\{v^x, v^y, v^z\}\) in \(S\); \(\{v'^{x}, v'^{y}, v'^{z}\}\) in \(S'\).

\[
v'^{x} = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx / c^2)} \times \frac{1}{dt} = \frac{dx}{dt} - \frac{v}{c^2} \frac{dx}{dt} = \frac{v^x - v}{1 - \frac{vv^x}{c^2}}
\]

\[
v'^{y} = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - vdx / c^2)} \times \frac{1}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \frac{v^y}{1 - \frac{vv^x}{c^2}}
\]

Inverses:

\[
v^x = \frac{v'^{x} + v}{1 + \frac{vv'^{x}}{c^2}} \quad \text{Note: Exchange primed and Unprimed variables and } v \text{ by } -v.
\]

\[
v^y = \sqrt{1 - \frac{v^2}{c^2}} \frac{v'^{y}}{1 + \frac{vv'^{x}}{c^2}}
\]