

# Quantum Many Body Theory

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Mathematical supplements

# 1 Analytic continuation

Analytic continuation is a technique in the theory of function of complex variables. It is based on the theorem that, in a somewhat simplified form, states that if you know some function for real values of its argument, you also know it for complex values of its argument. Here “know” has mathematical sense: the function of the complex variable, under these conditions, exists and is unique. How you find it is another story. Thus the technique of analytic continuation is about how to find it.

Let's illustrate it on the example of this integral:

$$I(z) = \int_{-\infty}^{\infty} \frac{dp}{p^2 + z}. \quad (1.1)$$

Here  $z$  is a complex variable. We know that this integral is convergent, thus it exists for any complex number  $z$ .

We will calculate this integral using the technique of the analytic continuation. First we will calculate it for real positive  $z = \rho$ , where  $\rho > 0$  is a real positive number. There are many ways to calculate this integral when  $z$  is real and positive, and the answer is

$$I(x) = \int_{-\infty}^{\infty} \frac{dp}{p^2 + \rho} = \frac{\pi}{\sqrt{\rho}}, \quad \rho > 0. \quad (1.2)$$

Let us use this result to calculate  $I(z)$  for complex  $z$ . The theorem guarantees that the answer for complex  $z$  is also  $\pi/\sqrt{z}$ . However, there is a problem. Square root is not a uniquely defined function.  $\sqrt{z}$  can take one of two possible values. For example, if

$$z = \rho e^{i\phi} \text{ where } \rho > 0 \quad (1.3)$$

(any complex number can be represented in this form), then

$$\sqrt{z} = \sqrt{\rho} e^{i\frac{\phi}{2}} \text{ or } \sqrt{z} = -\sqrt{\rho} e^{i\frac{\phi}{2}}. \quad (1.4)$$

The question is which of the two values of square root we should take to calculate the value of the integral (1.1) for some complex  $z$ .

We resolve this issue by analytic continuation. If  $\text{Im } z > 0$ , then let us first calculate this integral at  $z = \rho$ , and then change the phase  $\phi$  at the expression  $z = \rho e^{i\phi}$  from 0 to the appropriate positive phase  $0 < \phi < \pi$  (because, if  $\text{Im } z > 0$ , that's the appropriate range of  $\phi$ ). As the phase is changing we should use the expression

$$I(z) = \frac{\pi}{\sqrt{\rho} e^{i\frac{\phi}{2}}}, \quad 0 < \phi < \pi \quad (1.5)$$

and not  $-\sqrt{\rho}e^{i\phi/2}$ . The reason being that the integral  $I(z)$  cannot change discontinuously when  $z$  is changing continuously. We start at  $z = \rho$  and  $I(z) = \pi/\sqrt{\rho}$  and as  $z$  is changing continuously ( $\phi$  in  $z = \rho e^{i\phi}$  is increasing continuously), the integral has to change continuously. Thus it must be given by (1.5).

Now if  $\text{Im } z < 0$ , then  $z = \rho e^{i\phi}$  where  $0 > \phi > -\pi$ . Then we start at  $z = \rho$  and start decreasing  $\phi$ , all the while still using the expression

$$I(z) = \frac{\pi}{\sqrt{\rho}e^{i\frac{\phi}{2}}}, \quad 0 > \phi > -\pi \quad (1.6)$$

again for reasons of continuity.

(1.5) and (1.6) constitute the answer to our problem. Let us note though that if we use (1.5) to compute the value of this integral at  $\phi = \pi$ , or in other words  $z = \rho e^{i\pi} = -\rho$ , we find

$$I(-\rho) = \frac{\pi}{\sqrt{\rho}e^{i\pi/2}} = \frac{\pi}{i\sqrt{\rho}}. \quad (1.7)$$

On the other hand, if we represent  $z = -\rho = \rho e^{-i\pi}$ , then using (1.6)

$$I(-\rho) = \frac{\pi}{\sqrt{\rho}e^{-i\pi/2}} = \frac{\pi}{-i\sqrt{\rho}}. \quad (1.8)$$

We get two different answers!

The reason for the seeming contradiction is that the integral  $I(z)$  is not defined if  $z = -\rho < 0$ . It is in fact divergent, having a singularity at  $p = \pm\sqrt{\rho}$ . Analytic continuation is allowed as long as the integral is well defined. In fact,  $I(z)$  has a discontinuity if  $z$  changes from  $z = -\rho + i\epsilon$  to  $z = -\rho - i\epsilon$  due to this divergence. We can still use analytic continuation to calculate

$$\lim_{\epsilon \rightarrow 0, \epsilon > 0, \rho > 0} \int_{-\infty}^{\infty} \frac{dp}{p^2 - \rho + i\epsilon} \equiv \int_{-\infty}^{\infty} \frac{dp}{p^2 - \rho + i0} = \frac{\pi}{i\sqrt{\rho}}, \quad (1.9)$$

where (1.5) was used. Here  $i0$  is the notation we use throughout this class implying just this kind of limit.

On the other hand,

$$\lim_{\epsilon \rightarrow 0, \epsilon > 0, \rho > 0} \int_{-\infty}^{\infty} \frac{dp}{p^2 - \rho - i\epsilon} \equiv \int_{-\infty}^{\infty} \frac{dp}{p^2 - \rho - i0} = \frac{\pi}{-i\sqrt{\rho}}, \quad (1.10)$$

To obtain expression like (1.9) and (1.10) are what analytic continuation is often used for.

## 2 Delta-function

Delta-function is defined as a limit of functions

$$\delta(x) = \lim_{\epsilon \rightarrow 0} f_\epsilon(x). \quad (2.11)$$

Here  $f_\epsilon(x)$  obeys the following properties.

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} f_\epsilon(x) &= 0 & \text{if } x \neq 0, \\ \lim_{\epsilon \rightarrow 0} f_\epsilon(x) &= \infty & \text{if } x = 0, \\ \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dx f_\epsilon(x) &= 1. \end{aligned} \quad (2.12)$$

There exist many different  $f_\epsilon(x)$  which satisfy these properties. But they all define the same delta-function. One example is

$$f_\epsilon(x) = \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}. \quad (2.13)$$

Delta-function has many important properties. First of all,

$$\int_{-\infty}^{\infty} dx f(x) \delta(x - x_0) = f(x_0). \quad (2.14)$$

Second

$$\int_{-\infty}^{\infty} dx f(x) \delta(ax - x_0) = \frac{f(x_0/a)}{|a|}. \quad (2.15)$$

This formula can be derived by changing variables  $ax = y$  which gives

$$\frac{1}{|a|} \int_{-\infty}^{\infty} dy f(y/a) \delta(y - x_0) \quad (2.16)$$

and then using (2.14). Finally, an important formula which is a generalization of (2.15) says that in order to compute

$$\int_{-\infty}^{\infty} dx f(x) \delta(g(x) - x_0) \quad (2.17)$$

we need to first find solutions to the equation  $g(x) = x_0$ . Suppose these solutions are  $x_i$ . Then

$$\int_{-\infty}^{\infty} dx f(x) \delta(g(x) - x_0) = \sum_i \frac{f(g(x_i))}{|g'(x_i)|}. \quad (2.18)$$

This formula is derived by noting that delta-function is zero if its argument is nonzero. So only those values of  $x$  contribute where  $g(x) - x_0 = 0$ . Expanding  $g(x) - x_0$  in Taylor series

around  $x_i$  we find  $g(x) - x_0 = (x - x_i)g'(x_0)$  where the equality holds only for  $x$  being infinitesimally close to  $x_i$ . However, delta function is only nonzero if  $x$  is infinitesimally close to  $x_i$ . For each  $x \sim x_i$ , we apply (2.15) to arrive at (2.18).

Example: compute

$$\int_{-\infty}^{\infty} dk \delta(E - k^2). \quad (2.19)$$

There are two solutions to  $E = k^2$ ,  $k = \sqrt{E}$  and  $k = -\sqrt{E}$ . If  $g(x) = k^2$ , then  $|g'(\sqrt{E})| = 2\sqrt{E}$  for the first solution and  $|g'(-\sqrt{E})| = 2\sqrt{E}$  for the second solution. Thus

$$\int_{-\infty}^{\infty} dk \delta(E - k^2) = \frac{1}{2\sqrt{E}} + \frac{1}{2\sqrt{E}} = \frac{1}{\sqrt{E}}. \quad (2.20)$$

A useful formula states:

$$\frac{1}{x + i0} = -i\pi\delta(x) + \text{real stuff}. \quad (2.21)$$

This formula means the following

$$\int_{-\infty}^{\infty} dx \frac{f(x)}{x + i0} \equiv \lim_{\epsilon \rightarrow 0, \epsilon > 0} \int_{-\infty}^{\infty} dx \frac{f(x)}{x + i\epsilon} = -i\pi f(0) + \text{real stuff} \quad (2.22)$$

where real stuff depends on the function  $f(x)$  in some complicated way which is generally not known (but can be calculated for each specific  $f(x)$ ). However, it is known that it is real, not imaginary. Another way to write this formula

$$\text{Im} \int_{-\infty}^{\infty} dx \frac{f(x)}{x + i0} = -\pi f(0). \quad (2.23)$$

where Im denotes imaginary part.

One way to derive this formula is this. It is clear that

$$\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} = \frac{-2i\epsilon}{x^2 + \epsilon^2}. \quad (2.24)$$

Now it follows from (2.13) that

$$\lim_{\epsilon \rightarrow 0} \frac{-2i\epsilon}{x^2 + \epsilon^2} = -2i\pi\delta(x). \quad (2.25)$$

On the other hand, if  $1/(x + i0) = a_1(x) + ia_2(x)$ , where  $a_1$  and  $a_2$  are real functions, then

$$\frac{1}{x + i0} - \frac{1}{x - i0} = 2ia_2(x). \quad (2.26)$$

That's because  $1/(x - i0)$  is a complex conjugate of  $1/(x + i0)$ . Comparison with (2.25) gives

$$a_2(x) = -\pi\delta(x), \quad (2.27)$$

which proves (2.21). Notice that this technique leaves  $a_1(x)$  undetermined.

Let us apply this formula to the integral

$$\text{Im} \int_{-\infty}^{\infty} \frac{dp}{p^2 - \rho + i0} = -i\pi \int_{-\infty}^{\infty} dp \delta(p^2 - \rho) = -\frac{i\pi}{\sqrt{\rho}} \quad (2.28)$$

where (2.20) was used. This is consistent with (1.9)! Notice that this is only consistent, not equivalent. That is because (1.9) proves that the real part of this integral is actually zero, while (2.28) only computes the imaginary part of this integral, leaving the real part undetermined.

Equivalently, we find

$$\text{Im} \int_{-\infty}^{\infty} \frac{dp}{p^2 - \rho - i0} = +i\pi \int_{-\infty}^{\infty} dp \delta(p^2 - \rho) = \frac{i\pi}{\sqrt{\rho}} \quad (2.29)$$

consistent with (1.10). The formula  $1/(x - i0) = i\pi\delta(x) + \text{real stuff}$  is a complex conjugate of (2.21).

Finally, the last useful (and very famous and popular) formula states

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx}. \quad (2.30)$$

This formula can be derived for example like this:

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} = \lim_{\epsilon \rightarrow 0} \left[ \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx + p\epsilon} + \int_0^{\infty} \frac{dp}{2\pi} e^{ipx - p\epsilon} \right] = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \left[ \frac{-i}{px - i\epsilon} + \frac{i}{px + i\epsilon} \right] = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} = \delta(x), \quad (2.31)$$

where again (2.13) was used.

### 3 3D delta function potential

A delta-function potential in 3D, such as  $V(\vec{r}) = \lambda \delta(x)\delta(y)\delta(z)$  (where  $x, y, z$  are the three dimensional components of the vector  $\vec{r}$ ), is a badly defined object. Particles do not feel this potential at all, because it is too “narrow”, and fly straight through it. Realistic potentials in the world could be short ranged, but are not a delta function (after all, delta function is an abstraction).

A delta-function potential is very convenient in order to do calculations using Green's functions, so it is tempting to approximate a realistic short range potential by a delta function. But in 3D, this makes no sense, because delta function makes no sense!

To find a way out, we will use, instead of a 3D delta function, some "widened" delta function. For example,

$$\lambda \delta(x) \rightarrow \frac{\lambda}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}. \quad (3.32)$$

We have to do it with  $x$ ,  $y$ , and  $z$  part of the 3D delta function above, but for illustration reasons, we will explicitly write only the  $x$ -part.

This guy has a width, of the order of  $\epsilon$ , which we will keep "small" but finite. To be able to use it in calculations we need its matrix element

$$\langle k_f | \frac{\lambda}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} | k_i \rangle = \frac{\lambda \epsilon}{\pi} \int_{-\infty}^{\infty} dx \frac{e^{ik_i x - ik_f x}}{x^2 + \epsilon^2} = \lambda e^{-\epsilon |k_f - k_i|}. \quad (3.33)$$

Compare it with the matrix element of the delta function, which would've been just equal  $\lambda$ . For  $|k_f - k_i| \ll 1/\epsilon$ , this matrix element approximately coincides with the matrix element of the delta function. But for very large momenta difference,  $|k_f - k_i| \gg 1/\epsilon$ , this matrix element goes to zero exponentially.

Now in a typical calculation we need to compute things like

$$\int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \langle k_f | V | k_1 \rangle \langle k_1 | \hat{G} | k_2 \rangle \langle k_2 | V | k_i \rangle = \int \frac{dk_1}{2\pi} \langle k_f | V | k_1 \rangle G_0(k_1) \langle k_1 | V | k_i \rangle. \quad (3.34)$$

Here we used that  $\langle k_1 | \hat{G} | k_2 \rangle = 2\pi \delta(k_1 - k_2) G_0(k_1)$  to eliminate one of the integrals, over  $k_2$ , as discussed in the main text.

With  $V$  being the delta function we subsequently used the fact that the matrix element of  $V$  is  $k$ -independent if it is a delta function. Because of that, the integral over  $k_1$  is independent of  $k_f$  and of  $k_i$  and reduces to  $\int dk_1 G_0(k_1)$ . This same integral appears in all orders of the calculation, and that's what makes the problem solvable.

However, if we use (3.33), the integral over  $k_1$  becomes more complicated. And in fact, while it may in principle be doable, once we go to higher order terms, the appropriate integrals will become more and more complicated. We will never hope to compute them all and then resum the series, like we did for the delta function potential.

This is of course natural. Replacing the delta function by (3.32) we replaced a solvable potential to the one whose Schrödinger equation is much more complicated. There is no hope to solve this problem exactly using Green's functions.

On the other hand, in 3D we can't work with delta function - it is meaningless. What can we do?

A trick around it is called the “separable potential”. Imagine there exist a potential such that

$$\langle k_f | V(x) | k_i \rangle = \lambda e^{-\epsilon|k_i|} e^{-\epsilon|k_f|} \quad (3.35)$$

In practice such potential does not exist. But it mimics (3.33) in some sense. Then substituting into (3.34) we find

$$\lambda^2 e^{-\epsilon|k_f| - \epsilon|k_i|} \int \frac{dk_1}{2\pi} e^{-2\epsilon|k_1|} G_0(k_1). \quad (3.36)$$

The potential “separated” and its pieces got attached to the Green’s function to be integrated. Now we need to do the integral

$$\int \frac{dk_1}{2\pi} e^{-2\epsilon|k_1|} G_0(k_1). \quad (3.37)$$

This same integral will appear in all orders of the calculation, and the series can be subsequently resummed as a geometric series. So the problem of such separable potential can be solved.

For 3D delta-function, treated in this way, the appropriate integral will read

$$\int \frac{d^3 k_1}{(2\pi)^3} \frac{e^{-2\epsilon|k_1|}}{E + i0 - \frac{k_1^2}{2m}}. \quad (3.38)$$

We can do this integral in the following way.

$$\int \frac{k_1^2 dk_1}{2\pi^2} \frac{e^{-2\epsilon|k_1|}}{E + i0 - \frac{k_1^2}{2m}} = \int \frac{dk_1}{2\pi^2} \frac{e^{-2\epsilon|k_1|} (k_1^2 - 2mE)}{E + i0 - \frac{k_1^2}{2m}} + \int \frac{dk_1}{2\pi^2} \frac{2mE e^{-2\epsilon|k_1|}}{E + i0 - \frac{k_1^2}{2m}} \quad (3.39)$$

The integral is split into two integrals. The first one is

$$-2m \int_0^\infty \frac{dk_1}{2\pi} e^{-2\epsilon|k_1|}. \quad (3.40)$$

It is finite, can be computed, depends on  $\epsilon$  and does not depend on  $E$ .

The second integral is

$$\int \frac{dk_1}{2\pi^2} \frac{2mE e^{-2\epsilon|k_1|}}{E + i0 - \frac{k_1^2}{2m}} \quad (3.41)$$

This integral has a finite limit as  $\epsilon \rightarrow 0$ . Since  $\epsilon$  is small anyway, this is a legitimate limit, and it gives

$$\int \frac{dk_1}{2\pi^2} \frac{2mE}{E + i0 - \frac{k_1^2}{2m}}. \quad (3.42)$$

This integral can be computed in a closed form and depends on  $E$  but does not depend on  $\epsilon$ . Thus we found

$$\int \frac{d^3 k_1}{(2\pi)^3} \frac{e^{-2\epsilon|k_1|}}{E + i0 - \frac{k_1^2}{2m}} = \text{something which depends on } \epsilon + \text{something which depends on } E. \quad (3.43)$$

The crucial thing is that the part which depends on  $E$  is an interesting part, and it is not only independent of  $\epsilon$ , it is even independent of the precise form of the potential! If instead of  $e^{-\epsilon|k|}$  we used something else, like for example  $e^{-\epsilon^2 k^2}$  or  $1/(1 + \epsilon^2 k^2)$ , the result would be the same!

So in the main text where we worked with the 3D delta function we used  $\theta(\Lambda - |k_1|)$ , where  $\theta$  is the famous theta-function (equal to 1 if its argument is positive and to 0 if it is negative). We computed this integral

$$\int_{|k_1| \leq \Lambda} \frac{d^3 k_1}{(2\pi)^3} \frac{1}{E + i0 - \frac{k^2}{2m}}. \quad (3.44)$$

This integral also consists of two parts, one depends only on  $\Lambda$  (which plays the role of  $1/\epsilon$ ) while the other depends on  $E$ . The part which depends on  $E$  is exactly equal to (3.42), according to our discussions. This is the interesting physical part which does not depend on which "regularization", or precisely which separable potential, is being used.

If we now recall how the answer to the Green's function (actually, T-matrix) in a potential looks like,

$$\frac{1}{\frac{1}{\lambda} - \Pi}. \quad (3.45)$$

Here  $\Pi$  is the integral (3.37) we've been computing. This gives

$$\frac{1}{\frac{1}{\lambda} - f_1(\epsilon) - f_2(E)}. \quad (3.46)$$

In an actual experiment nobody measures  $\lambda$  or  $\epsilon$ . It is the T-matrix which is being measured. So we can denote  $1/\lambda - f_1(\epsilon) = 1/\lambda^*$  and use  $\lambda^*$  everywhere.  $\lambda^*$  can be measured experimentally, and it is the only physical parameter. In other words, all the details about which separable potential is being used disappeared. It does not matter which one one uses, the answer is the same.

The conclusion: replacing the delta function by a separable potential of a given "width" results in a solution which is basically insensitive to which potential, and of which width, is being used.

## 4 Properties of the many body fermionic Green's functions

### 4.1 Retarded Green's function

A retarded many-body function is defined by (in case of fermions)

$$G^R(x_f, x_i; t_f, t_i) = -i \left\langle \hat{\psi}(x_f, t_f) \hat{\psi}^\dagger(x_i, t_i) + \hat{\psi}^\dagger(x_i, t_i) \hat{\psi}(x_f, t_f) \right\rangle \theta(t_f - t_i). \quad (4.47)$$

Let us derive the Lehmann decomposition for this function. As is standard (see Week 5 of the notes), we replace the Heisenberg operators  $\hat{\psi}(x, t)$ ,  $\hat{\psi}^\dagger(x, t)$  by  $e^{i\hat{H}t - i\hat{P}x} \hat{\psi} e^{-i\hat{H}t + i\hat{P}x}$  and  $e^{i\hat{H}t - i\hat{P}x} \hat{\psi}^\dagger e^{-i\hat{H}t + i\hat{P}x}$  where  $\hat{\psi}$  and  $\hat{\psi}^\dagger$  are now Schrödinger operators located at the origin of the reference frame. Then we insert a complete set of states between  $\hat{\psi}$  and  $\hat{\psi}^\dagger$ . This leads to the equation

$$G^R(x, t) = \theta(t) \left[ -i \sum_n |\langle n | \hat{\psi}^\dagger | 0 \rangle|^2 e^{-i(\epsilon_n^+ - \mu)t + iP_n x} - i \sum_n |\langle n | \hat{\psi} | 0 \rangle|^2 e^{-i(\epsilon_n^- - \mu)t - iP_n x} \right] \quad (4.48)$$

Here

$$\epsilon_n^+ = E_n(N+1) - E_0(N) > \mu, \quad \epsilon_n^- = E_0(N) - E_n(N-1) < \mu, \quad (4.49)$$

(see (5.25) and (5.27) of the Week 5 notes),  $P_n$  being the momentum of the  $n$ -th state, and

$$t = t_f - t_i, \quad x = x_f - x_i. \quad (4.50)$$

Now let us Fourier transform this

$$G^R(p, E) = \int_0^\infty dt \int_{-\infty}^\infty dx G^R(x, t) e^{iEt - ipx} \quad (4.51)$$

Doing the Fourier transform with respect to the coordinate is easy and gives the delta-functions. The Fourier transform with respect to time exists only if  $\text{Im } E > 0$ , so that the integral over time in (4.51) is convergent. This gives

$$G^R(p, E) = (2\pi)^d \sum_n \left[ \frac{|\langle n | \hat{\psi}^\dagger | 0 \rangle|^2 \delta(P_n - p)}{E + i0 - (\epsilon_n^+ - \mu)} + \frac{|\langle 0 | \hat{\psi} | n \rangle|^2 \delta(P_n + p)}{E + i0 - (\epsilon_n^- - \mu)} \right]. \quad (4.52)$$

Again, this expression works only if  $\text{Im } E \leq 0$ . If  $\text{Im } E < 0$ , then this expression cannot be trusted (the integral which was used to compute it in (4.51) is divergent, and the function

$G^R(E, p)$  can only be found by analytically continuing (4.52) from the real axis down to the lower half plane).

We conclude from here that in the upper half plane the retarded Green's function has no singularities of any kind. At the same time, it can have singularities in the lower half plane.

Other properties of the retarded Green's function include:

1.  $\text{Im } G^R < 0$  (if  $E$  is real). Indeed, recall that  $\text{Im } 1/(x + i0) = -\pi\delta(x)$ , according to (2.21). We then apply this to (4.52) to find

$$\begin{aligned} \text{Im } G^R(p, E) = & -\pi(2\pi)^d \sum_n \left[ \left| \langle n | \hat{\psi}^\dagger | 0 \rangle \right|^2 \delta(P_n - p) \delta(E - \epsilon_n^+ + \mu) + \right. \\ & \left. \left| \langle 0 | \hat{\psi}^\dagger | n \rangle \right|^2 \delta(P_n + p) \delta(E - \epsilon_n^- + \mu) \right]. \end{aligned} \quad (4.53)$$

2. If  $E \rightarrow \infty$ , then  $G^R(p, E) \sim 1/E$  (for any complex  $E$ ), or more precisely,

$$\lim_{E \rightarrow \infty} G(E, p)E = 1. \quad (4.54)$$

To see this, it is best to introduce  $\hat{a}_p, \hat{a}_p^\dagger$ , which are Fourier transforms of  $\hat{\psi}(x)$  and  $\hat{\psi}^\dagger(x)$ , where in turn  $\hat{\psi}(x, t) = e^{i\hat{H}t}\hat{\psi}(x)e^{-i\hat{H}t}$  etc. That is, we do not introduce the factors of  $e^{-i\hat{P}x}$  as we did previously. Then at large  $E$  we neglect everything in denominators of (4.52) to find

$$G(p, E) \sim \frac{1}{E} \sum_n \left[ \left| \langle n | \hat{\psi}_p^\dagger | 0 \rangle \right|^2 + \left| \langle 0 | \hat{\psi}_p^\dagger | n \rangle \right|^2 \right]. \quad (4.55)$$

$$\begin{aligned} G(p, E) & \sim \frac{1}{E} \sum_n \left[ \left| \langle n | \hat{\psi}_p^\dagger | 0 \rangle \right|^2 + \left| \langle 0 | \hat{\psi}_p^\dagger | n \rangle \right|^2 \right] \\ & \frac{1}{E} \left( \sum_n \langle 0 | \hat{\psi}_p | n \rangle \langle n | \hat{\psi}_p^\dagger | 0 \rangle + \sum_n \langle 0 | \hat{\psi}_p^\dagger | n \rangle \langle n | \hat{\psi}_p | 0 \rangle \right) = \\ & \frac{1}{E} \langle 0 | \hat{\psi}_p \hat{\psi}_p^\dagger + \hat{\psi}_p^\dagger \hat{\psi}_p | 0 \rangle = \frac{1}{E}. \end{aligned} \quad (4.56)$$

3. The retarded Green's function has not only no poles if  $\text{Im } E > 0$ , but also it has no zeroes there. This is a rather subtle relation and follows from the Kramers-Kronig relations (which are true for any function which has no singularities in the upper half plane). See for example Landau and Lifshitz, statistical physics, volume 5, section 123 (the generalized susceptibility).

## 4.2 Time-ordered Green's function

A time ordered Green's function has been studied in the Week 5 of the notes (see Eq. (6.12 of those notes), with the result that its Lehmann decomposition is given by

$$G(p, E) = (2\pi)^d \sum_n \left[ \frac{|\langle n | \hat{\psi}^\dagger | 0 \rangle|^2 \delta(P_n - p)}{E + i0 - (\epsilon_n^+ - \mu)} + \frac{|\langle 0 | \hat{\psi}^\dagger | n \rangle|^2 \delta(P_n + p)}{E - i0 - (\epsilon_n^- - \mu)} \right]. \quad (4.57)$$

Here the first term in the square brackets has no singularities in the upper half plane, and the second term has no singularities in the lower half plane.

The properties of this Green's function include

1.  $\text{sign Im } G(E, p) = -\text{sign } E$ , if  $E$  is real. Indeed, analogously to (4.53)

$$\begin{aligned} \text{Im } G(p, E) = & -\pi(2\pi)^d \sum_n \left[ |\langle n | \hat{\psi}^\dagger | 0 \rangle|^2 \delta(P_n - p) \delta(E - \epsilon_n^+ + \mu) - \right. \\ & \left. |\langle 0 | \hat{\psi}^\dagger | n \rangle|^2 \delta(P_n + p) \delta(E - \epsilon_n^- + \mu) \right]. \end{aligned} \quad (4.58)$$

Then the sign of the imaginary part of  $G$  follows from this and from (4.49).

2.  $G(E, p) = G^R(E, p)$  if for complex  $E$  such that  $\text{Re } E > 0$ ,  $\text{Im } E > 0$ .  $G(E, p) = G^R(E^*, p) = G^{R*}(E, p)$  for complex  $E$  such that  $\text{Re } E < 0$ ,  $\text{Im } E < 0$ . Here star  $*$  denotes the complex conjugation. This is a rather nontrivial statement, and it follows from the following.

First take  $E$  real. Then it is clear that the imaginary part of  $G$  coincides with the imaginary part of  $G^R$  if  $E > 0$  (compare (4.53) and (4.58), not forgetting (4.49)). If  $E < 0$ , then the imaginary parts of  $G$  and  $G^R$  have opposite signs. The real parts of  $G$  and  $G^R$  coincide no matter what  $E$  is. Thus for real  $E < 0$ ,  $G^* = G^R$ . Analytically continue  $G^R$  to the upper half plane. This corresponds to the analytically continuing  $G$  to the upper half plane if  $\text{Re } E > 0$ , or to the lower half plane if  $\text{Re } E < 0$ . Which shows that indeed, the time-ordered and retarded Green's functions coincide in the manner described.

3. It follows from the previous statement that the Green's function  $G$  has no singularities in the upper half plane such that  $\text{Re } E > 0$ , and no singularities in the lower half plane such that  $\text{Re } E < 0$ . Indeed, combine 2 with the fact that  $G^R$  has no singularities of any kind in the upper half plane.

4.  $G(E) \sim 1/E$  at large  $E$ . The proof is the same as the similar statement for  $G^R$ .