

Quantum Many Body Theory

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Week 9

10 The Luttinger Theorem

The interacting gas, no matter how strong its interactions, obeys a theorem which says that (in case of a spin 1/2 gas)

$$\frac{N}{2V} = \frac{p_F^3}{6\pi^2}. \quad (10.1)$$

It relates the Fermi momentum to the total number of particles. The theorem was proved by Luttinger and Ward, and independently by Landau and Pitaevskii, around 1960. Let us present a proof of this theorem. The factor of 2 in $N/(2V)$ comes from the fact that we've counted only particles of one spin; counting both would multiply the equation by 2 on both sides.

The total number of particles is given by

$$\frac{N}{2V} = -i \lim_{t \rightarrow -0} \int \frac{dE d^3p}{(2\pi)^4} G(E, p) e^{-iEt}. \quad (10.2)$$

Recall the general relation

$$G(E, p) = \frac{1}{E - \frac{p^2}{2m} + \mu - \Sigma(E, p)}. \quad (10.3)$$

It follows that

$$G = -\frac{\partial}{\partial E} \log(G) + G \frac{\partial \Sigma}{\partial E}, \quad \frac{N}{V} = -i \lim_{t \rightarrow -0} \int \frac{dE d^3p}{(2\pi)^4} \left[-\frac{\partial}{\partial E} \log(G) + G \frac{\partial \Sigma}{\partial E} \right] e^{-iEt}. \quad (10.4)$$

The first step in the proof is to prove that

$$\int \frac{dE}{2\pi} G \frac{\partial \Sigma}{\partial E} = 0. \quad (10.5)$$

This is proven by showing that there exists a functional, called the *Luttinger functional*, $X[G]$ such that

$$\Sigma(E, p) = \frac{\delta X}{\delta G(E, p)}. \quad (10.6)$$

If this is indeed so, then we can take the integral by parts

$$\int \frac{dE d^3p}{(2\pi)^4} G \frac{\partial \Sigma}{\partial E} = - \int \frac{dE d^3p}{(2\pi)^4} \frac{\partial G}{\partial E} \Sigma = - \int \frac{dE d^3p}{(2\pi)^4} \frac{\delta X}{\delta G(E, p)} \frac{\partial G}{\partial E} = 0. \quad (10.7)$$

The boundary term, when taking this integral by parts, vanishes because $G \sim 1/E$, and Σ grows slower than E , as was discussed during the problem session 2. The last equality follows from the fact that

$$\frac{\partial}{\partial a} X[G(E + a, p)] = 0 = \int dE d^3p \frac{\delta X}{\delta G(E + a, p)} \frac{\partial G(E + a, p)}{\partial a}. \quad (10.8)$$

So now all that is left is to show the existence of such X , in order to prove (10.5). To do that, it's enough to prove that

$$\frac{\delta\Sigma(E, p)}{\delta G(E', p')} = \frac{\delta^2 X}{\delta G(E, p) \delta G(E', p')} \quad (10.9)$$

is symmetric under the simultaneous interchange of E with E' , and p with p' .

Σ consists of all possible one particle irreducible diagrams. $\delta\Sigma/\delta G$ are all such diagrams with one propagator removed. All such diagrams together constitute a vertex function at the incoming energy-momenta E, P and E', p' and the same outgoing energy-momenta. Such vertex function is of course symmetric under the particle exchange as described. This concludes the proof of (10.5). Thus we arrive at

$$\frac{N}{2V} = i \lim_{t \rightarrow -0} \int \frac{dE d^3p}{(2\pi)^4} \frac{\partial \log(G)}{\partial E} e^{-iEt}. \quad (10.10)$$

The integral over E seems simple because it is an integral over total derivative. Yet one needs to remember that the presence of the exponential prefactor is important, and the integral needs to be over the contour which closes in the upper half plane. In particular, if G is a free Fermi gas Green's function, $G = 1/(E - p^2/(2m) + \mu + i0 \text{ sign } E)$, then $\log(G)$ changes by $-2\pi i$ along this contour if $p^2/(2m) < \mu$, and by 0 if $p^2/(2m) > \mu$, leading to $n_0(p)$ to be integrated over p and leading to the standard relation between N/V and p_F .

In the more general case of the arbitrarily interacting gas, we note that G always has the singularities in the lower half plane if $\text{Re } E > 0$ and in the upper half plane if $\text{Re } E < 0$. Define a function $G^R(E) = G(E)$ for $E > 0$ and $G^R(E) = G^*(E)$ if $E < 0$. This function G^R has no poles in the upper half plane (in fact, it coincides with the retarded Green's function as follows from the Lehmann decomposition, see lecture notes of Week 5 and the mathematical supplements). Then

$$\frac{N}{2V} = i \oint \frac{dE}{2\pi} \frac{d^3p}{(2\pi)^3} \frac{\partial \log(G^R)}{\partial E} + i \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^0 \frac{dE}{2\pi} \log \left[\frac{G(E)}{G^*(E)} \right] \quad (10.11)$$

The first integral is over the contour which is closed in the upper half plane, and it has no poles there. So it is zero. (Notice that the logarithm has singularity not only if its argument has a pole, but also if its argument is equal to zero. However, G^R cannot be equal to zero in the upper half plane. See mathematical supplements and Landau and Lifshitz, volume 5, section 123).

If we denote the phase of the function $G(E)$ by $\phi(E)$, then

$$-\frac{1}{\pi} \int \frac{d^3p}{(2\pi)^3} [\phi(0) - \phi(-\infty)]. \quad (10.12)$$

At $-\infty$, we know that $G(E) \sim 1/E$. This means that its imaginary part is much smaller than its real part there, since $G(E)$ asymptotically approaches a real value. At the same time, $G(E) < 0$ at minus infinity. Thus, $\phi(-\infty) = \pi$. Since, as we know, $\text{Im } G > 0$ if $E < 0$ and $\text{Im } G = 0$ if $E = 0$ (follows from the Lehmann decomposition, lecture notes of Week 5), then ϕ can only vary between 0 and π as a function of E . It is clear that $\phi(0)$ is equal to π if $G(0) < 0$ and to 0 if $G(0) > 0$. Thus it follows that

$$\frac{N}{2V} = \int_{G(0,p) > 0} \frac{d^3p}{(2\pi)^3}. \quad (10.13)$$

The region $G(0,p) > 0$ in momentum space is bounded by a surface where G is either 0 or infinity. We expect that G is infinity on the Fermi surface (that defines the Fermi momentum). Indeed, this is the point where the poles of the Green's function occur at $E = 0$, thus corresponding to gapless (technical term for zero energy) excitations. Then the integral over the area where $G_0(0,p) > 0$ gives us

$$\frac{N}{2V} = \frac{p_F^3}{6\pi^2}, \quad (10.14)$$

which completes the proof.

There is a substantial interest in this theorem, even nowadays. It follows from it that a Fermi gas always has a Fermi surface (except in cases of superconductivity) of a certain fixed volume, so it is interesting to try to understand what this implies for the more “exotic” states of fermions, such as the topological states of matter observed in quantum Hall effect and in other situations. See, for example, M. Oshikawa, Phys. Rev. Lett. 84, 3370 (2000).