Quantum Many Body Theory

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Week 8

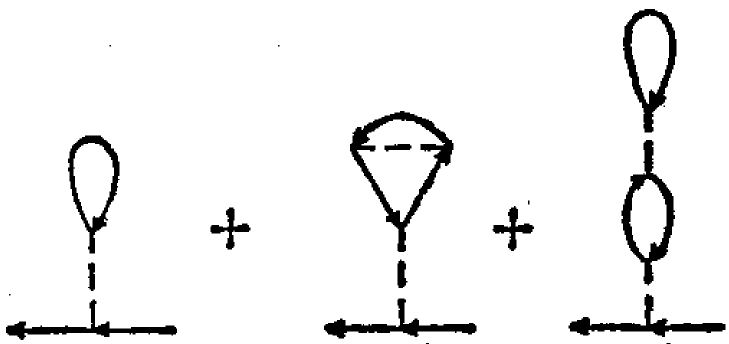

Figure 1: Diagrams up to the second order contributing to the self energy which are energy-momentum independent

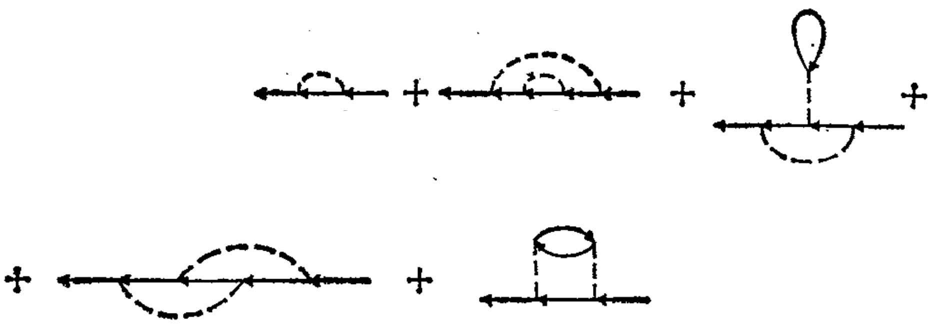

Figure 2: The rest of the self energy diagrams up to the second order

9 Weakly Interacting Fermi Gas

We would like to calculate the interaction correction to the Green's function of the Fermi gas. We assume that the gas is weakly interacting, or the scattering length a of two-particle scattering problem is much smaller than the particle separation l. The range of the potential r_0 is also much smaller than l (and in fact, for a repulsive potential, $r_0 \leq a$). The figures 1 and 2 are all possible diagrams which contribute to the self energy.

These notes closely follow Landau and Lifshits, Statistical Physics Part 2, (Lifshitz and Pitaevskii), Section 21.

9.1 First order corrections

In the lowest order of perturbation theory, there are only two 1PI diagrams. They are shown on Figs. 3 and 4. We will denote all first order corrections to Σ by $\Sigma^{(1i)}$, where $i = 1, 2$ are two possible first order diagrams. The sum of all the first order corrections will be denoted as $\Sigma^{(1)}$.

The diagram shown on Fig. 3 is the simplest. It includes a loop made of a Green's function. This loop is nothing but $\left\langle \psi^\dagger(x,t)\psi(x,t) \right\rangle$ which is the density of a free Fermi gas (computed as a function of the chemical potential μ). Then it also includes the integral over the point x of $U(x-y)$, which gives the zero Fourier component of the potential which we denote as $U(0)$ (not to be confused with $U(x)|_{x=0}$, of course).

Thus this diagram gives the first contribution to the self-energy, denoted $\Sigma^{(1)}$, or

$$\Sigma^{(11)} = U(0) n_0(\mu), \tag{9.1}$$

where $n_0(\mu) = (4/3)\pi(2m\mu)^{\frac{3}{2}}/(2\pi)^3$ is the density of the free Fermi gas.

The next diagram shown on Fig. 4 is in fact as simple. It includes an integral over a loop made by the potential and the Green's function. The potential is time independent, thus only the Green's function depends on the frequency and that can be integrated out to give $n_0(p)$, that is, the particle number at momentum p. Then the remaining integral is thus

$$\int d^3p \, n(p) U(Q-p), \tag{9.2}$$

where Q is the momentum going though the diagram. However, the potential is short ranged. This means that it deviates from $U(0)$ only at very large momenta of the order of

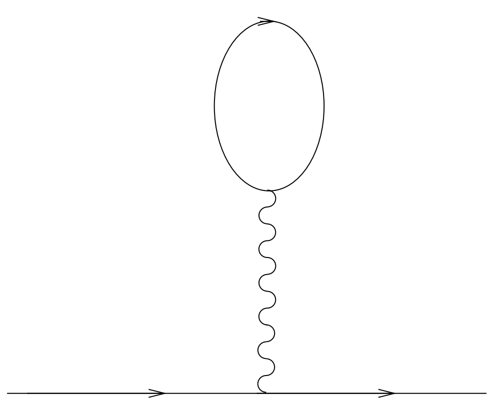

Figure 3: The first self-energy diagram.

$1/r_0$ which is much bigger than p_F. Thus this diagram gives the same thing as (9.1), but half as big and with the opposite sign, or

$$\Sigma^{(12)} = -\frac{1}{2}U(0)n_0(\mu). \tag{9.3}$$

Together these give

$$\Sigma^{(1)} = \frac{1}{2}U(0)n_0(\mu). \tag{9.4}$$

This concludes the calculation of the first order corrections to the Green's functions. The self-energy at this order is momentum and energy independent constant.

The Green's function at this order of approximation to Σ is given by

$$G = \frac{1}{E - \frac{p^2}{2m} + \mu - \frac{1}{2}U(0)n_0(\mu) + i0 \text{ sign } E}. \tag{9.5}$$

This gives, for the number of particles

$$n(\mu) = \frac{p_F^3}{3\pi^2},\ \frac{p_F^2}{2m} = \mu - \frac{1}{2}U(0)n_0(\mu). \tag{9.6}$$

Here $n_0(\mu)$ is the particle number computed in the *free* Fermi gas, or $n_0(\mu) = (2m\mu)^{3/2}/(3\pi^2)$. We can solve this equation for μ, considering $U(0)$ a small parameter. This gives, after some algebra

$$\mu = \frac{(3\pi^2 n)^{\frac{2}{3}}}{2m}\left(1 + \frac{U(0)mn^{\frac{1}{3}}}{(3\pi^2)^{\frac{2}{3}}}\right). \tag{9.7}$$

The scattering amplitude in the lowest Born approximation is $f = -U(0)m/(4\pi)$, and on the other hand, $f = -a$ where a is the scattering length. Thus

$$\mu = \frac{p_F}{2m}\left(1 + \frac{4ap_F}{3\pi}\right). \tag{9.8}$$

We see that the true small parameter is $ap_F \ll 1$, and this is the expansion in powers of it. This is the beginning of the expansion of μ in powers of $p_F a$ first computed by Yang and Huang in 1957 (they went up to terms $(p_F a)^2$ though). Knowing μ as a function of n allows us to compute the ground state energy because

$$\frac{\partial E_0}{\partial N} = \mu. \tag{9.9}$$

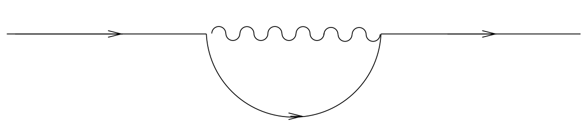

Figure 4: The second self-energy diagram

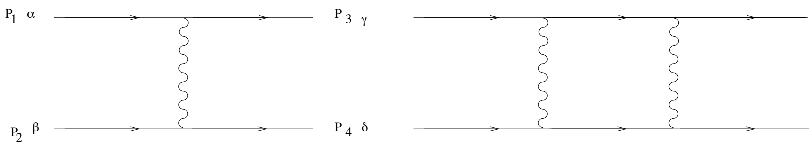

Figure 5: The two intermediate diagrams

Thus

$$E = N\frac{3p_F^2}{10m}\left(1 + \frac{10}{9\pi}p_F a\right). \tag{9.10}$$

9.2 Second order corrections

The second order corrections are more interesting. They correct not only the energy and chemical potential, but also the particle spectrum (because Σ is no longer momentum and energy independent).

Two such corrections are given by the same diagrams shown on Figs. 3 and 4, but with the Green's function (the one which forms a loop into itself in Fig. 3 or the one which forms a loop together with the interaction line in Fig. 4) corrected itself by the first order corrections. These are, precisely, diagrams 2 and 3 on Fig. 1 and diagrams 2 and 3 on Fig. 2. This can be summarized by saying that

$$\Sigma^{(1)} + \Sigma^{(21)} = \frac{1}{2}U(0)n(\mu), \tag{9.11}$$

where $n(\mu)$ is a more precise particle density at a given chemical potential (computed not for a free gas, but for an interacting gas in the first order of perturbation theory). That is, given by the expression (9.6).

The two additional diagrams are new in this order. These are two last diagrams on Fig. 2 (the one on the second line on this figure). They are the only nontrivial diagrams at this order, and can be calculated in the following way.

We introduce notations, where capital P means three dimensional vector $\mathbf{p}$ and the energy to be denoted p_0.

As an intermediate step, we introduce the following two diagrams, shown on Fig. 5. The second of these two diagrams form the basis for the diagrams we would like to compute. Indeed, both diagrams on the second line of Fig. 2 can be thought of as the second diagram on Fig. 5 with lines connected in some way. The first diagram shown on Fig. 5 is given by (remember that in our convention, the vertex function is defined as iF)

$$iF^{(1)}_{\alpha,\beta;\gamma,\delta}(P_4, P_3; P_1, P_2) = -i\delta_{\alpha\gamma}\delta_{\beta\delta}U(P_3 - P_1). \tag{9.12}$$

and

$$iF^{(2)}_{\alpha,\beta;\gamma,\delta}(P_4,P_3;P_1,P_2)=\int G_0(P')U(P_1-P')G_0(P_1+P_2-P')U(P'-P_3)\frac{d^4P'}{(2\pi)^4}. \quad (9.13)$$

Now the two diagrams from the second line of Fig. 2 can be expressed in terms of $F^{(2)}$, by means of

$$-i\Sigma^{(22)}(P)=-\int G_0(Q)F^{(2)}(P,Q;Q,P)\frac{d^4Q}{(2\pi)^4}+2\int G_0(Q)F^{(2)}(P,Q;P,Q)\frac{d^4Q}{(2\pi)^4}. \quad (9.14)$$

Next step is to calculate $F^{(2)}$. This amounts to computing the integral

$$\int\frac{d\omega}{2\pi}\frac{1}{\omega-\frac{p'^2}{2m}+\mu+i0\,\mathrm{sign}\,(\omega)}\frac{1}{E_1+E_2-\omega-\frac{(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}')^2}{2m}+\mu+i0\,\mathrm{sign}\,(E_1+E_2-\omega)}. \quad (9.15)$$

This integral is different from zero only if the poles of the two Green's functions lie in the opposite half planes, or if

$$\mathrm{sign}\,(p'-p_F)=\ \mathrm{sign}\,(|\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}')|-p_F). \quad (9.16)$$

The integral thus gives

$$F^{(2)}=-\int\frac{d^3p'}{(2\pi)^3}\frac{U(\mathbf{p}_1-\mathbf{p}')U(\mathbf{p}'-\mathbf{p}_3)\,(1-n_0(\mathbf{p}')-n_0(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}'))}{E_1+E_2+2\mu-\frac{p'^2+(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}')^2}{2m}+i0\,\mathrm{sign}\,(p'-p_F)} \quad (9.17)$$

Here we defined a function $n_0(p)=(1-\ \mathrm{sign}\,(p-p_F))/2$ is the occupation function of the noninteracting fermions, equal to 1 if the momentum lies below the Fermi surface, and to zero otherwise. The expression in the integral forces the condition (9.16).

We would like to trade the potential for the scattering amplitude. Working in the second order of the perturbation theory, we replace $U(\mathbf{p}-\mathbf{q})$ by $-4\pi f/m$ where f is the scattering amplitude computed up to the second order (in other words, using (5) in vacuum. We do it by replacing U everywhere by the scattering length $a=-f(0)$, simultaneously subtracting from $F^{(2)}$ the expression for the second order correction to the scattering amplitude. This gives

$$\begin{aligned}F^{(2)} &= -\left(\frac{4\pi a}{m}\right)^2\int\frac{d^3p'}{(2\pi)^3}\left\{\frac{U(\mathbf{p}_1-\mathbf{p}')U(\mathbf{p}'-\mathbf{p}_3)\,(1-n_0(\mathbf{p}')-n_0(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}'))}{E_1+E_2+2\mu-\frac{p'^2+(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}')^2}{2m}+i0\,\mathrm{sign}\,(p'-p_F)}\right.\\ &\left.-\frac{2m}{p_1^2+p_2^2-(\mathbf{p_1}+\mathbf{p_2}-\mathbf{p}')^2}\right\}. \end{aligned} \quad (9.18)$$

Now we need to substitute this into

$$-i\Sigma = \int G_0(Q)F^{(2)}(P,Q;P,Q)\frac{d^4Q}{(2\pi)^4}. \quad (9.19)$$

where it was taken into account that $F^{(2)}$ is symmetric under the exchange of P_1 and P_2, after replacing the potential by the scattering amplitude. We can then do integral over the energy component of Q. Finally, the result is

$$\Sigma^{(2)} = \frac{2\pi}{m}n(\mu)a + \Sigma^{(22)}, \quad (9.20)$$

where

$$\Sigma^{(22)}(\omega,p) = \left(\frac{4\pi a}{m}\right)^2 \int \frac{d^3p'd^3q}{(2\pi)^6}\left\{\frac{[1-n_0(\mathbf{p}')-n_0(\mathbf{p}+\mathbf{q}-\mathbf{p}')]\,[n_0(\mathbf{q})-n_0(\mathbf{p}')]}{\omega+\mu+\frac{1}{2m}\left[q^2-p'^2-(\mathbf{p}+\mathbf{q}-\mathbf{p}')^2\right]+i0\,\mathrm{sign}\,(p'-p_F)} - \frac{2mn_0(q)}{p^2+q^2-p'^2+(\mathbf{p}+\mathbf{q}-\mathbf{p}')^2+i0}\right\}. \quad (9.21)$$

We will not calculate the the integrals entering this expression, although it is not impossible (see Galitskii, Sov. Phys. JETP, **7**, 104 (1958), and also compare with problem 3, Problem Set 3).

We just comment that the imaginary part of this, when computed at $\omega = p^2/(2m) - \mu$, gives the lifetime of fermionic quasiparticles (compare with problem 1 of the problem set 3). That lifetime is nonzero only if

$$\frac{p^2+q^2-p'^2-(\mathbf{p}+\mathbf{q}-\mathbf{p}')^2}{2m} = 0. \quad (9.22)$$

This is but the energy conservation condition that a quasiparticle with the momentum p can decay into a hole of momentum q and the two electrons of momentum p' and $p+q-p'$. This leads to a finite lifetime of the particles.

The factors depending on n_0 simply counts the available states. Indeed, the decay is possible only if the hole state is initially filled, while the rest of the states are empty, and the reverse processes when the hole state is empty while the rest are filled also decrease the decay, and we see that

$$[1-n_0(\mathbf{p}')-n_0(\mathbf{p}+\mathbf{q}-\mathbf{p}')]\,[n_0(\mathbf{q})-n_0(\mathbf{p}')] = n_0(q)(1-n_0(p'))(1-n_0(\mathbf{p}+\mathbf{q}-\mathbf{p}'))-(1-n_0(q))n_0(p')n_0(\mathbf{p}+\mathbf{q}-\mathbf{p}'). \quad (9.23)$$