Quantum Many Body Theory

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Week 7

8 Particle Interactions

We now concentrate on the behavior of the interacting Fermi gas. Typical fermions are electrons, and they are spin 1/2 particles. So we will add spin, so that the Hamiltonian becomes

$$\hat{H} = \frac{1}{2m} \int dx \sum_{\alpha=\uparrow,\downarrow} \frac{\partial \hat{\psi}^\dagger_\alpha}{\partial x} \frac{\partial \hat{\psi}_\alpha}{\partial x} + \frac{1}{2} \sum_{\alpha,\beta=\uparrow,\downarrow} \int dx dy \, \hat{\psi}^\dagger_\alpha(x) \hat{\psi}_\alpha(x) U(x-y) \hat{\psi}^\dagger_\beta(y) \hat{\psi}_\beta(y) - \mu \hat{N}. \tag{8.1}$$

This equation implies that the interaction is between the fermions regardless of the value of their spin. Here $\hat{N} = \sum_{\alpha=\uparrow,\downarrow} \int dx \; \hat{\psi}^\dagger_\alpha(x) \hat{\psi}_\alpha(x)$. It is also common to write down the interactions as

$$\frac{1}{2} \sum_{\alpha,\beta=\uparrow,\downarrow} \int dx dy \, \hat{\psi}^\dagger_\alpha(x) \hat{\psi}^\dagger_\beta(y) U(x-y) \hat{\psi}_\beta(y) \hat{\psi}_\alpha(x) \tag{8.2}$$

Also notice the order of the operators which is chosen so that there is no minus signs, even though the operators are fermionic.

Interactions between the particles are best studied using the four-point Green's function. It is defined as

$$K_{\alpha,\beta;\gamma,\delta} = \left\langle T \hat{\psi}_\alpha(X_1) \hat{\psi}_\beta(X_2) \hat{\psi}^\dagger_\gamma(Y_1) \hat{\psi}^\dagger_\delta(Y_2) \right\rangle \tag{8.3}$$

where X_1, X_2, Y_1, Y_2 are both coordinates and time.

In the zero order perturbation theory, this is equal to (by Wick's theorem)

$$K_{\alpha,\beta;\gamma,\delta} = \delta_{\alpha,\gamma} \delta_{\beta,\delta} \, G_0(X_1 - Y_1) G_0(X_2 - Y_2) - \delta_{\alpha,\delta} \delta_{\beta,\gamma} \, G_0(X_1 - Y_2) G_0(X_2 - Y_1). \tag{8.4}$$

More generally, one can write

$$\begin{aligned} K \;=\; & \delta_{\alpha,\gamma} \delta_{\beta,\delta} \, G_0(X_1 - Y_1) G_0(X_2 - Y_2) - \delta_{\alpha,\delta} \delta_{\beta,\gamma} \, G_0(X_1 - Y_2) G_0(X_2 - Y_1) + \\ & i \int dX_1' dX_2' dY_1' dY_2' \, G_0(X_1 - X_1') G_0(X_2 - X_2') G_0(Y_1' - Y_1) G_0(Y_2' - Y_2) \times \\ & \Gamma_{\alpha,\beta;\gamma,\delta}(X_1', X_2', Y_1', Y_2') \end{aligned} \tag{8.5}$$

This is the definition of Γ, which is sometimes called vertex function.

8.1 Vacuum scattering

Let us show that in vacuum, Γ is the scattering amplitude. For definiteness, we will consider scattering between one spin up and one spin down particle.

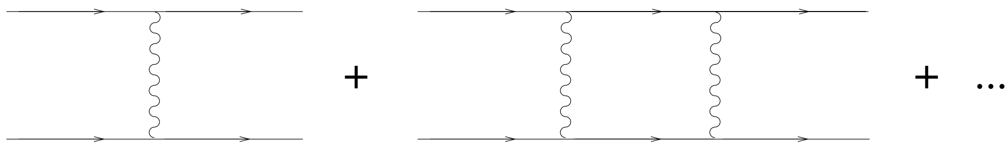

Figure 1: The diagrams which contribute to Γ in vacuum.

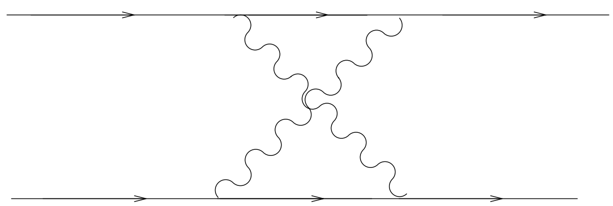

Figure 2: This diagram is zero in vacuum.

In vacuum, all the noninteracting Green's functions are retarded and given by

$$G_0 = \frac{1}{E - \frac{p^2}{2m} + i0}. \tag{8.6}$$

So the only diagrams which contribute to Γ are shown on Fig. 1. On the other hand, diagrams such as the ones shown on Fig. 2 are zero. Indeed, the interactions occur instantaneously in time, and we can find a closed loop where each point along the loop is ahead in time compared to the previous one.

The first diagram shown on Fig. 1 gives

$$\Gamma^{(0)}_{\uparrow,\downarrow;\uparrow,\downarrow}(X_1, X_2, Y_1, Y_2) = U(X_1 - X_2)\delta(X_1 - Y_1)\delta(X_2 - Y_2). \tag{8.7}$$

In the Fourier space, we can write

$$\Gamma^{(0)}_{\uparrow,\downarrow;\uparrow,\downarrow}(Q_1, Q_2; P_1, P_2) = U(P_1 - P_3)\,(2\pi)^4\delta(Q_2 + Q_1 - P_2 - P_1). \tag{8.8}$$

where P_1 and P_2 are two incoming momenta (plus frequency), and Q_3, Q_4 are the two outgoing momenta. Suppose we now calculate the second order diagram. See Fig. 3 where all the lines are labelled by their momenta. The energy labeling is similar. The loop in the diagram corresponds to the integral over energy (the interaction lines are energy independent, because they are delta functions in time domain and thus their Fourier transform with respect to time is constant)

$$i\int \frac{d\omega}{\left(E/2 + \omega - \frac{(p/2+q)^2}{2m} + i0\right)\left(E/2 - \omega - \frac{(p/2-q)^2}{2m} + i0\right)} = \frac{1}{E - \frac{q^2}{m} - \frac{p^2}{4m}} \tag{8.9}$$

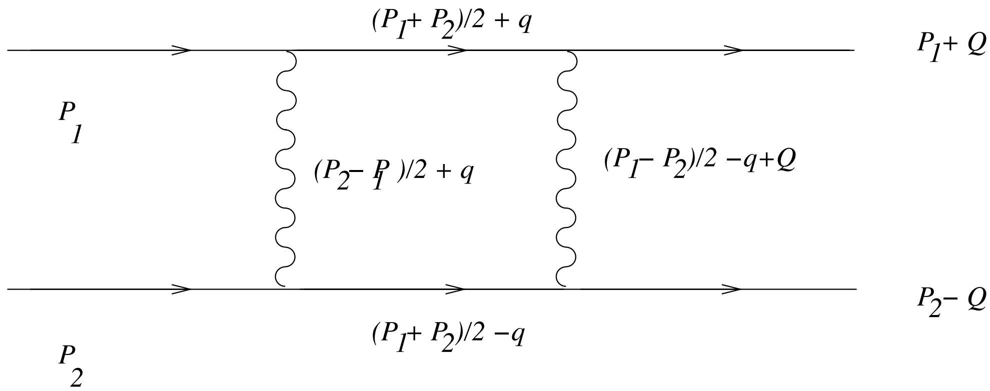

Figure 3: The second order diagram with momenta labelled.

Here $E = E_1 + E_2$, and $p = p_1 + p_2$ where $P_1 = (E_1, p_1)$ and $P_2 = (E_2, p_2)$. Thus the diagram shown on Fig. 3 can be thought of as

$$\int dq\, U(q_1 - q) \frac{1}{E - \frac{q'^2}{m} - \frac{p^2}{4m}} U(q - q_2). \tag{8.10}$$

Here $q_1 = (p_1 - p_2)/2$ and $q_2 = (p_1 - p_2)/2 + Q$. For the moment, Q means just the momentum, not the momentum and the energy.

More generally, we get a sum corresponding to the diagrams shown on Fig. 1,

$$U(q_1 - q_2) + \int dq\, U(q_1 - q) \frac{1}{E - \frac{q'^2}{m} - \frac{p^2}{4m}} U(q - q_2) + \dots \tag{8.11}$$

This coincides with the series we had for a T-matrix of a particle in vacuum, whose energy is $E - p^2/(4m)$ (see Weeks 1 and 2). The particle has mass $m/2$ (reduced mass of two particles), and the energy is reduced by $p^2/(4m)$ because the center of mass frame moves with respect to the frame where the calculation is performed.

Suppose these particles form a bound state. Then the mass of this bound state is $2m$ as can be seen from the denominator of the Green's function in (8.11), in its p dependence.

The conclusion is thus: the vertex function in vacuum coincides with a T-matrix of a "reduced" particle of a mass $m/2$ moving in an "external" potential given by the two body potential of the interacting problem.