Quantum Many Body Theory

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Week 6

7 Feynman Diagrams

7.1 Interaction representation

To compute Green's functions perturbatively, Feynman used the formalism of functional integrals. Back in late 40s few people appreciated how useful it was (nowadays this is the default method). Pretty soon however Dyson invented a way to construct perturbation theory without functional integrals. The technique relies on the interaction representation of the operators.

Suppose the Hamiltonian of a system is

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{7.1}$$

where $\hat{H}_0$ is a free Hamiltonian and $\hat{V}$ is interaction (what is implied here is that $\hat{H}_0$ is quadratic in creation and annihilation operators and thus exactly solvable).

We introduce the time dependence of the operators using

$$\hat{A}_0(t) = e^{i\hat{H}_0 t}\hat{A}e^{-i\hat{H}_0 t}, \tag{7.2}$$

that is, it is a Heisenberg representation with $\hat{H}_0$ instead of $\hat{H}$ as a Hamiltonian. This is called the "interaction representation".

To find the wave function dependence we write the average of some operator $\hat{A}$ over a time dependent wave function $e^{-i\hat{H}t}\Psi$ as

$$\left\langle e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}\right\rangle = \left\langle e^{i\hat{H}t}e^{-i\hat{H}_0 t}\hat{A}_0(t)e^{i\hat{H}_0 t}e^{-i\hat{H}t}\right\rangle \tag{7.3}$$

Thus the wave function in the interaction representation looks like

$$\Psi_0(t) = e^{i\hat{H}_0 t}e^{-i\hat{H}t}\Psi. \tag{7.4}$$

In other words,

$$i\frac{\partial \Psi_0}{\partial t} = e^{i\hat{H}_0 t}\left(-\hat{H}_0 + \hat{H}\right)e^{-i\hat{H}t}\Psi = e^{i\hat{H}_0 t}\hat{V}e^{-i\hat{H}t}\Psi = e^{i\hat{H}_0 t}\hat{V}e^{-i\hat{H}_0 t}\Psi_0. \tag{7.5}$$

The wave function in the interaction representation satisfies the Schrödinger equation, where the role of the Hamiltonian is played by the interaction term only, modified by the free term according to

$$i\frac{\partial \Psi_0}{\partial t} = \hat{V}_0(t)\Psi_0(t),\ \hat{V}_0(t) = e^{i\hat{H}_0 t}\hat{V}e^{-i\hat{H}_0 t}\Psi_0. \tag{7.6}$$

7.2 Examples of the interaction representation

To get a feel what the interaction representation means, consider this familiar Hamiltonian (see Week 3, Eq. (3.16))

$$\hat{H} = \frac{1}{2m} \int dx \frac{\partial \hat{\psi}^\dagger}{\partial x} \frac{\partial \hat{\psi}(x)}{\partial x} + \frac{1}{2} \int dx dx' \, \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') U(x - x') \hat{\psi}(x) \hat{\psi}(x'). \tag{7.7}$$

In the momentum (Fourier) space this looks like

$$\hat{H} = \sum_k \frac{k^2}{2m} \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2V} \sum_{P,q,q'} \hat{a}_{P+q}^\dagger \hat{a}_{P-q}^\dagger U(q' - q) \hat{a}_{P+q'} \hat{a}_{P-q'}. \tag{7.8}$$

(V is the volume, not interactions). Then the interaction in the interaction representation looks like

$$\hat{V}_0(t) = \frac{1}{2V} \sum_{P,q,q'} \hat{a}_{P+q}^\dagger \hat{a}_{P-q}^\dagger U(q' - q) \hat{a}_{P+q'} \hat{a}_{P-q'} \, e^{i \frac{(P+q)^2}{2m} t} e^{i \frac{(P-q)^2}{2m} t} e^{-i \frac{(P+q')^2}{2m} t} e^{-i \frac{(P-q')^2}{2m} t}. \tag{7.9}$$

(compare with (5.8), Week 5).

7.3 Green's functions via interaction representation

Suppose we would like to compute the Green's function

$$G(x - x', t - t') = -i \langle \Psi | T \hat{\psi}(x, t) \hat{\psi}^\dagger(x', t') | \Psi \rangle \tag{7.10}$$

in a problem governed by the Hamiltonian (7.1). The first step is to rewrite (7.10) in terms of the interaction representation.

We define an operator $\hat{S}(t_f, t_i)$ such that the solution to (7.6) reads

$$\Psi_0(t_f) = \hat{S}(t_f, t_i) \Psi_0(t_i). \tag{7.11}$$

It is tempting to say that $\hat{S}(t_f, t_i) = e^{-i\hat{V}_0(t_f)(t_f - t_i)}$, but of course this is incorrect. If $\hat{V}$ was time independent, this would be correct, but with $\hat{V}_0$ depending on time, it is easy to see that this does not work. In fact, we know what does work, it is (7.4). So we can write

$$\hat{S}(t_f, t_i) = e^{i\hat{H}_0(t_f - t_i)} e^{-i\hat{H}(t_f - t_i)}. \tag{7.12}$$

Since $\hat{H}_0$ and $\hat{H}$ do not commute we cannot combine the exponentials in (7.4) in any meaningful way and reduce the expressions in (7.12) to something like $e^{-i\hat{V}(t_f - t_i)}$.

Next step is to rewrite (7.10) in terms of the operators in the interaction representation. This is achieved by (let us for definiteness assume that $t > t'$ so we can omit the time order expression for now)

$$\begin{aligned} G(x-x',t-t') &= -i \langle\Psi| e^{i\hat{H}t}\hat{\psi}(x)e^{-i\hat{H}t}e^{i\hat{H}t'}\hat{\psi}^\dagger(x')e^{-i\hat{H}t'} |\Psi\rangle = \\ & -i \langle\Psi| e^{i\hat{H}t}e^{-i\hat{H}_0t}\hat{\psi}_0(x)e^{i\hat{H}_0t}e^{-i\hat{H}t}e^{i\hat{H}t'}e^{-i\hat{H}_0t'}\hat{\psi}_0^\dagger(x')e^{i\hat{H}_0t}e^{-i\hat{H}t'} |\Psi\rangle \\ & -i \langle\Psi| \hat{S}(0,t)\hat{\psi}_0(x,t)\hat{S}(t,t')\hat{\psi}_0^\dagger(x',t')\hat{S}(t',0) |\Psi\rangle \end{aligned} \tag{7.13}$$

This equation assumes that $t > t'$. If $t < t'$ we get the same expression, but with $\hat{\psi}$ operators exchanged. Most importantly, $\hat{S}$ operators always go towards increasing time, except the very last (leftmost) $\hat{S}$ operator. This equation also assumes that the initial moment in time was 0. It is more common to assume that the initial moment in time was at $-\infty$. Thus we arrive at the following expression

$$G(x-x',t-t') = -i \langle\Psi| \hat{S}(-\infty,t)\hat{\psi}_0(x,t)\hat{S}(t,t')\hat{\psi}_0^\dagger(x',t')\hat{S}(t',-\infty) |\Psi\rangle , \tag{7.14}$$

again assuming that $t > t'$. For completeness, let us write down the expression if $t < t'$:

$$G(x-x',t-t') = -i \langle\Psi| \hat{S}(-\infty,t')\hat{\psi}_0^\dagger(x,t')\hat{S}(t',t)\hat{\psi}_0(x',t)\hat{S}(t,-\infty) |\Psi\rangle , \tag{7.15}$$

The next step is to find $\hat{S}$ in terms of $\hat{V}_0(t)$. This is done by solving (7.6) by the method of successive approximations. We write

$$\Psi_0(t) = \Psi + \Psi^{(1)}(t) + \Psi^{(2)}(t) + \dots \tag{7.16}$$

Then we find

$$\Psi^{(1)}(t) = \int_{-\infty}^{t} dt'\, \hat{V}_0(t')\Psi. \tag{7.17}$$

The second order approximation reads

$$\Psi^{(2)}(t) = \int_{-\infty}^{t} dt'\, \hat{V}_0(t') \int_{-\infty}^{t'} dt''\, \hat{V}_0(t'')\Psi = \frac{1}{2}\int_{-\infty}^{t} dtdt'\, T\hat{V}_0(t)\hat{V}_0(t')\Psi. \tag{7.18}$$

A general term reads

$$\Psi^{(n)}(t) = \frac{1}{n!} T \int_{-\infty}^{t} \prod_{i=1}^{n} dt_n \hat{V}_0(t_n)\Psi. \tag{7.19}$$

This is written in the following way

$$\hat{S}(t_f,t_i) = Te^{-i\int_{t_i}^{t_f} dt\, \hat{V}_0(t)}. \tag{7.20}$$

Finally, writing $\hat{S}(-\infty,t)=\hat{S}(-\infty,\infty)\hat{S}(\infty,t)$, we can write the following expression corresponding to the Green's function (valid at both $t>t'$ and $t<t'$)

$$G(x-x',t-t')=-i\langle\Psi|\,\hat{S}(-\infty,\infty)T\left[\hat{\psi}_0(x,t)\hat{\psi}_0^\dagger(x',t')\hat{S}(+\infty,-\infty)\right]|\Psi\rangle\,. \tag{7.21}$$

This expression is perfectly suitable for expansion in powers of $\hat{V}_0(t)$ which is the perturbation. One little thing remains, what is Ψ? It should be the ground state of the interacting system, which we do not know. So we employ a trick: we say suppose interactions were zero in the infinite past and then slowly they turned on. Then Ψ is the ground state of the *noninteracting* system, which we know very well.

If the interactions were turning on very slowly, then the ground state evolves adiabatically. Assuming the interactions also turn off in the infinite future, we find that $\langle\Psi|\,\hat{S}(-\infty,\infty)=\langle\Psi|\,e^{i\phi}$, where ϕ is some phase. This allows us to write

$$G(x-x',t-t')=-i\langle\Psi|\,\hat{S}(-\infty,\infty)\,|\Psi\rangle\,\langle\Psi|\,T\left[\hat{\psi}_0(x,t)\hat{\psi}_0^\dagger(x',t')\hat{S}(+\infty,-\infty)\right]|\Psi\rangle\,. \tag{7.22}$$

The most common way of writing this expression is

$$G(x-x',t-t')=-i\frac{\langle\Psi|\,T\left[\hat{\psi}_0(x,t)\hat{\psi}_0^\dagger(x',t')\hat{S}(+\infty,-\infty)\right]|\Psi\rangle}{\langle\Psi|\,\hat{S}(\infty,-\infty)\,|\Psi\rangle}. \tag{7.23}$$

This expression, together with

$$\hat{S}(\infty,-\infty)=Te^{-i\int_{-\infty}^{\infty}dt\,\hat{V}_0(t)}. \tag{7.24}$$

forms the basis of the perturbative expansion scheme. The idea is to expand the exponential in (7.24) to compute the Green's function perturbatively in powers of the interactions.

7.4 Wick's theorem and Feynman diagrams

Once one expands the exponential, one starts getting high order "correlation functions", such as

$$\left\langle T\prod_i\hat{\psi}_0(x_i,t_i)\prod_i\hat{\psi}_0^\dagger(x_j,t_j)\right\rangle \tag{7.25}$$

The Wick's theorem states that this is equal to

$$\left\langle T\prod_i\hat{\psi}_0(x_i,t_i)\prod_i\hat{\psi}_0^\dagger(x_j,t_j)\right\rangle=\sum_{\text{permutations of }m_j}\prod_{i,j}\left\langle T\hat{\psi}_0(x_i,t_i)\hat{\psi}_0^\dagger(x_{m_j},t_{m_j})\right\rangle, \tag{7.26}$$

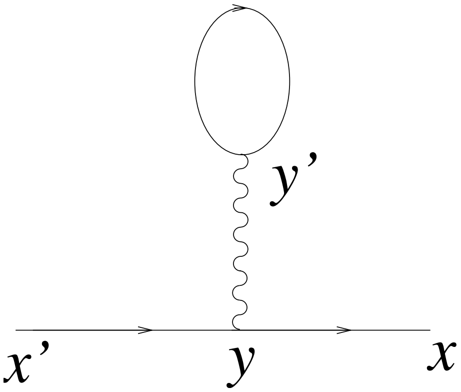

Figure 1: The simplest electronic self energy diagram. The interactions are represented by a wavy line, while the particles are represented by straight lines.

where m_j are various permutations. Finally,

$$G_0 = \left\langle \psi_0(x,t)\psi_0^\dagger(x',t') \right\rangle \tag{7.27}$$

is nothing but the Green's function of the noninteracting particles, found earlier.

Using Wick's theorem, we can construct terms in the perturbative expansion. For example, suppose we expand (7.23) once in powers of $\hat{V}_0(t)$. Suppose we work with our usual Hamiltonian (7.7). We find

$$\begin{aligned} \delta G &= -i \left\langle T\hat{\psi}_0(x,t)\hat{\psi}_0^\dagger(x',t') \left(-\frac{i}{2}\right) \times \right. \\ & \left. \int dy dy' dt'' dt''' \hat{\psi}_0^\dagger(y',t''')\psi_0^\dagger(y,t'')U(y-y')\delta(t''-t''')\psi_0(y,t'')\psi_0(y',t''') \right\rangle \end{aligned} \tag{7.28}$$

The equation is written in somewhat unconventional way [with the potential being $U(y-y')\delta(t''-t''')$ to emphasize symmetry between space and time]. In principle one can also talk about potential which is not delta-function in time. We can use Wick's theorem now in many different ways. One way is to connect x' to y, y to x, while to connect y' to itself. Taking care of the factors of i, we can write

$$\delta G = -i \int dy dy' dt'' dt''' G_0(x,t;y,t'')G_0(y,t'';x',t')U(y-y')\delta(t''-t''')G_0(y',t''';y',t'''). \tag{7.29}$$

The factor of 1/2 disappeared because of the symmetry between y and y'. This expression is represented by the diagram shown on Fig 1. We call this δG to emphasize that this is the correction to G in the first order of perturbation theory.

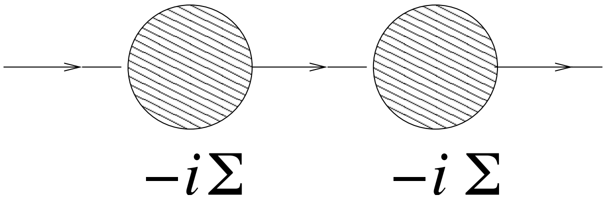

Figure 2: A Green's function contains an arbitrary number of self-energy diagrams

7.5 Feynman rules

A diagram is a pictorial way to write expressions corresponding to terms of the perturbation series. Once the diagram is drawn, we write its expression following these rules.

1. Each vertex implies a factor of $-i$. Each vertex sits at a point in space and time, and its coordinates are integrated over
2. Each line is a free Green's function iG_0.
3. Each diagram has an overall numerical factor corresponding to the number of ways one can connect its lines and get the same diagram.
4. If the particles are fermions, each closed loop contributes a factor of -1.
5. Wavy lines represent the interaction (or sometimes particles of a different kind)

It is often easier to do calculations in momentum space. Then instead of integrating over coordinates of each vertex, we introduce momentum and energy propagating along each line. Momentum and energy flowing though closed loops need to be integrated over.

7.6 Self energy diagrams

All diagrams which cannot be cut into two halves by cutting just one line are called one particle irreducible (1PI) diagrams. An arbitrary Green's function consists of lines connecting these 1IP diagrams, as shown on Fig. 2. The sum of all such diagrams is denoted by $-i\Sigma$. The factor $-i$ is of course a convention so that the self-energy diagrams would look like an effective "vertex".

In the momentum space the diagram such as the one depicted on Fig 2 corresponds to $iG_0(-i\Sigma)iG_0(-i\Sigma)iG_0$, where these are just literal products (no integrals involved). It follows that the sum of all such expressions with arbitrary number of Σ is

$$G = G_0 \sum_{n=0}^{\infty} (-i\Sigma i G_0)^n = \frac{1}{G_0^{-1} - \Sigma}. \tag{7.30}$$

This expression is exact. In other words, it is enough to know Σ to calculate the full (interacting) Green's function G.

In the literature, the formula (7.30) is sometimes written in the form

$$G = G_0 + G_0 \Sigma G, \tag{7.31}$$

and in this form it is called the Dyson equation.