Quantum Many Body Theory

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Week 4

4 Applications of the creation and annihilation operators

4.1 A tight binding model

Consider bosonic particles which hops around on the lattice, for simplicity, in 1D. Its Hamiltonian can be written as

$$\hat{H} = -t \sum_m \left(\hat{a}^\dagger_m \hat{a}_{m+1} + \hat{a}^\dagger_{m+1} \hat{a}_m \right). \tag{4.1}$$

m labels the sites of the lattice. This is called the tight-binding model in the literature. We can solve it by employing the unitary (Fourier) transformation

$$\hat{b}_q = \frac{1}{\sqrt{N}} \sum_m \hat{a}_m \, e^{-imq}, \;\; \hat{a}_m = \frac{1}{\sqrt{N}} \sum_q \hat{b}_q \, e^{imq}. \tag{4.2}$$

Here N is the number of sites of the lattice. Substituting this into the Hamiltonian, we find

$$\hat{H} = -2 \sum_q \cos(q) \, \hat{b}^\dagger_q \hat{b}_q. \tag{4.3}$$

This is a bunch of particles moving with (quasi) momentum q and with the energy

$$\epsilon_q = -2 \, \cos(q). \tag{4.4}$$

4.2 Holstein-Primakoff bosons

Suppose we have a particle of spin S. It is described by its spin operators, $\hat{S}^z$, $\hat{S}^+ = \hat{S}^x + i\hat{S}^y$, and $\hat{S}^- = \hat{S}^x - i\hat{S}^y$, which have the following commutation relations:

$$[\hat{S}^z, \hat{S}^+] = \hat{S}^+, \;\; [\hat{S}^z, \hat{S}^-] = \hat{S}^-, \;\; [\hat{S}^+, \hat{S}^-] = 2\hat{S}^z. \tag{4.5}$$

It is well known that there are $2S+1$ states representing the projection of the spin on the z-axis, which we denote $|S\rangle$, $|S-1\rangle$, $|S-2\rangle$, …, $|-S\rangle$. $\hat{S}^z$, when acting on these states, gives that projection. For example,

$$\hat{S}^z \left| S-1 \right\rangle = (S-1) \left| S-1 \right\rangle. \tag{4.6}$$

The idea of Holstein and Primakoff is to replace the spin operators by creation and annihilation operators. This can be done in the following way

$$\hat{S}^z = S - \hat{b}^\dagger \hat{b}, \;\; \hat{S}^+ = \sqrt{2S - \hat{b}^\dagger \hat{b}} \; \hat{b}, \;\; \hat{S}^- = \hat{b}^\dagger \sqrt{2S - \hat{b}^\dagger \hat{b}}. \tag{4.7}$$

Here the bosons represent the deviation of the spin from its maximum S^z value of S. If $S^z = S$, that means zero bosons. $S^z = S - 1$ means one boson. And so on, until $S^z = -S$ means $2S$ bosons. There can be no more bosons, because the square roots in (4.7) would make no sense. Raising and lowering spin operators are like bosons annihilation and creation operators.

The expressions such as $\sqrt{2S - \hat{b}^\dagger \hat{b}}$ might look intimidating. But because they only involve the $\hat{b}^\dagger \hat{b}$ combination, which simply counts bosons, it is an extremely straightforward operator. For example, since

$$\hat{b}^\dagger \hat{b} \left| S_z \right\rangle = (S - S_z) \left| S_z \right\rangle , \tag{4.8}$$

it follows that

$$\sqrt{2S - \hat{b}^\dagger \hat{b}} \left| S_z \right\rangle = \sqrt{S + S_z} \left| S_z \right\rangle . \tag{4.9}$$

It is straightforward to prove that (4.7) indeed works provided that $[\hat{b}, \hat{b}^\dagger] = 1$, that is, these guys are bosons. For example,

$$[\hat{S}^z, \hat{S}^+] = -[\hat{b}^\dagger \hat{b}, \sqrt{2S - \hat{b}^\dagger \hat{b}}\ \hat{b}] = \sqrt{2S - \hat{b}^\dagger \hat{b}}\ \hat{b} = \hat{S}^+. \tag{4.10}$$

This is so because applying $\hat{b}^\dagger \hat{b}$ (which counts the number of bosons) after applying that object which includes $\hat{b}$ and some square root, will always give answer which is bigger by 1 compared to what you get if you apply $\hat{b}^\dagger \hat{b}$ before. That's because $\hat{b} \cdot \sqrt{\dots}$ reduces number of particles by 1. Equivalently,

$$[\hat{S}^z, \hat{S}^-] = -[\hat{b}^\dagger \hat{b}, \hat{b}^\dagger \sqrt{2S - \hat{b}^\dagger \hat{b}}] = -\hat{b}^\dagger \sqrt{2S - \hat{b}^\dagger \hat{b}} = -\hat{S}^-. \tag{4.11}$$

Finally,

$$[\hat{S}^+, \hat{S}^-] = [\hat{b}\sqrt{2S - \hat{b}^\dagger \hat{b}}, \hat{b}^\dagger \sqrt{2S - \hat{b}^\dagger \hat{b}}] = \sqrt{2S - \hat{b}^\dagger \hat{b}} \left(1 + \hat{b}^\dagger \hat{b}\right) \sqrt{2S - \hat{b}^\dagger \hat{b}} - \hat{b}^\dagger \left(2S - \hat{b}^\dagger \hat{b}\right) \hat{b}. \tag{4.12}$$

In the first term everything commutes, and in the second $\hat{b}$ can be the moved to the left at the expense of $\hat{b}^\dagger \hat{b}$ decreasing in value, to give

$$\left(1 + \hat{b}^\dagger \hat{b}\right) (2S - \hat{b}^\dagger \hat{b}) - \hat{b}^\dagger \hat{b}(2S - \hat{b}^\dagger \hat{b} + 1) = 2S - 2\hat{b}^\dagger \hat{b} = 2S_z. \tag{4.13}$$

This completes the proof.

It is often important to study just small deviations of the spin from the upward position, in cases when spin is very long or $S \gg 1$. Then $\hat{b}^\dagger \hat{b} \ll 2S$, and we can approximately write

$$S_z = 2S - \hat{b}^\dagger \hat{b},\ S^+ = \hat{b}\sqrt{2S},\ S^- = \hat{b}^\dagger \sqrt{2S}. \tag{4.14}$$

4.3 The spectrum of a ferromagnet

Consider a system of spins interacting ferromagnetically

$$\hat{H} = -J \sum_m \vec{\hat{S}}_m \vec{\hat{S}}_{m+1} = -J \sum_k \left[\hat{S}^z_m \hat{S}^z_{m+1} + \frac{1}{2} \left(\hat{S}^+_m \hat{S}^-_{m+1} + \hat{S}^-_m \hat{S}^+_{m+1} \right) \right]. \tag{4.15}$$

The lowest energy spin configuration corresponds to all spins pointing in the same direction. The excitations correspond to spins deviating from the same direction slightly. We can use (4.7) to find

$$\hat{H} = 2SJ \sum_m \hat{b}^\dagger_m \hat{b}_m - JS \sum_m \left(\hat{b}^\dagger_{m+1} \hat{b}_m + \hat{b}^\dagger_m \hat{b}_{m+1} \right). \tag{4.16}$$

This is similar to the tight binding model (4.1) we studied before. Introducing the Fourier transformed variables

$$\hat{a}_q = \frac{1}{\sqrt{N}} \sum_m \hat{b}_m \, e^{-imq}, \; \hat{b}_m = \frac{1}{\sqrt{N}} \sum_q \hat{a}_q \, e^{imq}. \tag{4.17}$$

we find

$$\hat{H} = SJ \sum_q \left(2 - 2\cos(q) \right) \hat{a}^\dagger_q \hat{a}_q. \tag{4.18}$$

In other words, this spin system behaves like a bunch of bosons (called magnons) with the wave vector q and the energy $\epsilon(q) = JS(2 - 2\cos(q))$. Notice that for small q

$$\epsilon(q) \approx JSq^2. \tag{4.19}$$

The spectrum is quadratic, like for free particles.

4.4 The spectrum of an antiferromagnet

Consider a system of spins interacting antiferromagnetically

$$\hat{H} = J \sum_m \vec{\hat{S}}_m \vec{\hat{S}}_{m+1} = J \sum_k \left[\hat{S}^z_m \hat{S}^z_{m+1} + \frac{1}{2} \left(\hat{S}^+_m \hat{S}^-_{m+1} + \hat{S}^-_m \hat{S}^+_{m+1} \right) \right]. \tag{4.20}$$

To minimize their energy, the spins want to alternate: one spin points up, the next one down, the next one up and so on. The fluctuations of a spin pointing up can be modeled by the linearized Holstein-Primakoff bosons, as before,

$$\hat{S}^z = S - \hat{b}^\dagger \hat{b}, \; \hat{S}^+ = \hat{b}\sqrt{2S}, \; \hat{S}^- = \hat{b}^\dagger \sqrt{2S}. \tag{4.21}$$

The fluctuations of a spin pointing down should be described differently. Indeed, now the Holstein-Primakioff should describe the deviations of S^z from $-S$ instead of S. We can do it by

$$\hat{S}^z = -S + \hat{b}^\dagger \hat{b}, \ \hat{S}^+ = \hat{b}^\dagger \sqrt{2S}, \ \hat{S}^- = \hat{b}\sqrt{2S}. \tag{4.22}$$

Substituting these into (4.20) gives

$$\hat{H} = 2JS \sum_m \hat{b}^\dagger_m \hat{b}_m + JS \sum_m \left(\hat{b}_m \hat{b}_{m+1} + \hat{b}^\dagger_m \hat{b}^\dagger_{m+1} \right). \tag{4.23}$$

Now we apply the Fourier transform (4.17)

$$\hat{a}_q = \frac{1}{\sqrt{N}} \sum_m \hat{b}_m \, e^{-imq}, \ \hat{b}_m = \frac{1}{\sqrt{N}} \sum_q \hat{a}_q \, e^{imq}, \tag{4.24}$$

to find

$$\hat{H} = 2JS \sum_q \hat{a}^\dagger_q \hat{a}_q + JS \sum_q \cos(q) \left(\hat{a}_q \hat{a}_{-q} + \hat{a}^\dagger_q \hat{a}^\dagger_{-q} \right). \tag{4.25}$$

Now we see that we encountered a situation discussed before: a quadratic Hamiltonian, but the one which includes not only products of creation and annihilation operators, but also squares of creation and annihilation operators. This situation needs to be attacked by an appropriate Bogoliubov transformation (see Problem 3 from Problem Set 1).

We write

$$\hat{a}_q = \hat{c}_q \cosh(\phi_q) - \hat{c}^\dagger_{-q} \sinh(\phi_q), \ \hat{a}^\dagger_q = \hat{c}^\dagger_q \cosh(\phi_q) - \hat{c}_{-q} \sinh(\phi_q). \tag{4.26}$$

We need to write $\hat{c}^\dagger_{-q}$ to be added to $\hat{c}_q$ because $\hat{c}_q$ and $\hat{c}^\dagger_{-q}$ have the same q-dependence.

Substitution of these into (4.25) requires quite a bit of algebra. Collecting all the terms proportional to $\hat{c}_q \hat{c}_{-q}$ gives

$$\sum_q \hat{c}_q \hat{c}_{-q} \left[-2\cosh(\phi_q)\sinh(\phi_q) + \cos q \left(\sinh(\phi_q)^2 + \cosh(\phi_q)^2 \right) \right]. \tag{4.27}$$

We want this term to be zero, that is, the stuff in the square brackets needs to vanish. This gives

$$-2\cosh(\phi_q)\sinh(\phi_q) + \cos q \left(\sinh(\phi_q)^2 + \cosh(\phi_q)^2 \right) = 0 \ \rightarrow \ \tanh(2\phi_q) = \cos(q). \tag{4.28}$$

This automatically also kills the terms $\hat{c}^\dagger_q \hat{c}^\dagger_{-q}$.

Now collecting all the terms proportional to $\hat{c}^\dagger_q \hat{c}_q$ gives

$$\hat{H} = 2JS \sum_q |\sin(q)| \, \hat{c}^\dagger_q \hat{c}_q. \tag{4.29}$$

Thus we find that the excitations of an antiferromagnet are bosons, with the spectrum

$$\epsilon(q) \sim |\sin(q)| \approx |q|. \tag{4.30}$$

Unlike those of a ferromagnet (4.19), these excitations have linear spectrum.