Quantum Many Body Theory

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Week 12

12 Motion in a random potential

12.1 Feynman diagrams

Consider a particle which moves in a potential $V(x)$,

$$\hat{H} = -\frac{1}{2m}\frac{d^2}{dx^2} + V(x). \tag{12.1}$$

Suppose $V(x)$ varies randomly in space (but remains constant in time). It is clear that understanding the motion of such a a particle would then be very difficult, if not impossible, to solve. Instead of trying to solve this problem for a particular realization of the random potential $V(x)$, one tries to compute averages over different realizations of the random potential.

Suppose on the average the potential is zero, $\langle V \rangle = 0$. Suppose that the two-point correlation of the potential is given by

$$\langle V(x)V(y)\rangle = U\delta(x-y). \tag{12.2}$$

We say that this is a short ranged correlated potential. (In practice, we choose the correlator to be delta-function for calculational convenience, but the result should not be sensitive to the range of correlations as long as it is much shorter than the wavelength of the particle, or $1/\sqrt{2mE}$, where E is the energy of the particle.

Suppose also that the potential is Gaussian. What this means is, the averages of products of many potentials is given by the Wick's theorem:

$$\left\langle \prod_i^{2n} V(x_i) \right\rangle = U^n\left[\delta(x_1-x_2)\delta(x_3-x_4)\ldots + \delta(x_1-x_3)\delta(x_2-x_4)\ldots + \ldots\right]. \tag{12.3}$$

This is a little harder to justify, but at the very least it substantially simplifies the calculations.

Then we can calculate average Green's functions, by using the familiar perturbative approach

$$\begin{aligned} G(x_f-x_i) = \quad & \langle x_f|\,\frac{1}{E-\hat{H}_0-\hat{V}}\,|x_i\rangle = G_0(x_f-x_i) + \int dx_1\, G_0(x_f-x_1)V(x_1)G_0(x_1-x_i) + \\ & \int dx_1 dx_2\, G_0(x_f-x_1)V(x_1)G_0(x_1-x_2)V(x_2)G_0(x_2-x_i) + \ldots. \end{aligned} \tag{12.4}$$

Let us average this over the random potential using Wick's theorem. We find

$$\langle G\rangle = G_0(x_f-x_i) + \int dx_1 dx_2\, G_0(x_f-x_1)G_0(x_1-x_2)U\delta(x_1-x_2)G_0(x_2-x_i) + \ldots \tag{12.5}$$

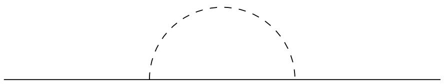

Figure 1: The simplest diagram, corresponding to (12.5). Here the dotted line is the potential correlation function $U\delta(x_1 - x_2)$.

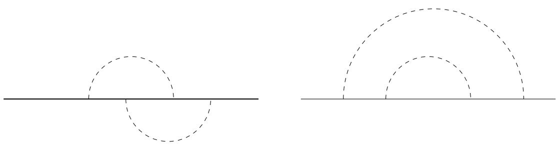

Figure 2: The second order 1PI diagrams.

The process of averaging is easy to illustrate using Feynman diagrams. Thus the process of constructing diagrams is clear. Notice that there is no factors of i anywhere - this is the time independent formalism, so no energy integration and no i.

12.2 Random matrices

Suppose the Hamiltonian is a completely random hermitian $N \times N$ matrix, H_{ij}, $H^\dagger = H$. On the average, it's zero, and its correlator is

$$\langle H_{ji} H_{kl} \rangle = \left\langle H^*_{ij} H_{kl} \right\rangle = h\, \delta_{ik} \delta_{jl}. \tag{12.6}$$

This problem was first suggested by Wigner to study spectra of nuclei. Presumably their Hamiltonians are so complicated that we don't know what they are, but maybe they can be modeled by a completely random matrix. This program turned out to be a success story in a certain branch of nuclear and particle physics.

Let us compute the average of the (retarded) Green's function,

$$G = \frac{1}{E + i0 - H}. \tag{12.7}$$

Expanding in powers of E, we generate the diagrams as usual (in what follows $E + i0$ will be replaced by E to save space, but it should be understood as $E + i0$).

$$G_{ij} = \frac{\delta_{ij}}{E} + \sum_{kl} \frac{\delta_{ik}}{E} h_{kl} \frac{\delta_{lj}}{E} + \dots \tag{12.8}$$

It is straightforward to see that some diagrams will be proportional to N^n, N being the size of the matrix and n is the order of the diagram. But some will only be proportional to N^{n-1} or even smaller powers of N. The largest diagrams proportional to N^n are called the "rainbow diagrams" (for example, the second diagram on Fig. 2). They are the largest, and they lead to the following equation for the self energy

$$\Sigma_{kl}(E) = h\delta_{kl} \sum_{i=1}^{N} G_{ii}, \; G_{ij} = \frac{\delta_{ij}}{E - \Sigma(E)}. \tag{12.9}$$

Here the parametrization $\Sigma_{ij} = \Sigma\delta_{ij}$ is introduced. This equation is called the "self-consistent Born approximation". This is a very common approximation. In this particular case, this approximation becomes exact if $N \to \infty$. We find

$$\Sigma = \frac{hN}{E - \Sigma} \;\to\; \Sigma^2 - E\Sigma + hN = 0. \tag{12.10}$$

Solving the quadratic equation, we find

$$\Sigma = \frac{E \pm \sqrt{E^2 - 4hN}}{2}. \tag{12.11}$$

If $E^2 < 4hN$, then the square root is purely imaginary. If we are looking at the retarded Green's function, $\text{Im}\,\Sigma < 0$, and we need to choose the appropriate sign. This gives

$$G = \frac{2\delta_{ij}}{E + i\sqrt{4hN - E^2}}. \tag{12.12}$$

This concludes the calculation of the Green's function in the self-consistent Born approximation (which is exact in the $N \to \infty$ (large matrices) limit).

Typically one is interested in the probability of observing an eigenvalue of the random matrix at some value E. This is given by the density of states,

$$\rho(E) = \frac{1}{N} \sum_n \delta(E - E_n) = -\frac{1}{\pi N} \text{Tr Im}\, G = \frac{1}{2\pi h N} \sqrt{4hN - E^2}. \tag{12.13}$$

This is the famous Wigner's semicircle law.

12.3 Particle in a random potential: self-consistent Born approximation

Let's try to apply the same approximation to the particle in a random potential (we'll discuss its applicability in this setting later). We find the analog of (12.10)

$$\Sigma(E, p) = U \int \frac{d^3q}{(2\pi)^3} \frac{1}{E + i0 - \frac{q^2}{2m} - \Sigma(E, p)}. \tag{12.14}$$

It's obvious from this equation that Σ does not depend on the momentum (but might depend on the energy).

The integral in (12.14) has been computed in the Week 2 notes. It's divergent and has to be cut off at large momenta $p \leq \Lambda$ (physically this means that the interaction is not correlated to a delta function, but rather to a short ranged function whose spacial extend is of the order of $1/\Lambda$).

$$\Sigma = U \left[-\frac{m\Lambda}{\pi^2} - \frac{m^{\frac{3}{2}}\sqrt{\Sigma - E}}{\pi\sqrt{2}} \right]. \quad (12.15)$$

We solve it by successive approximations. First, we take

$$\Sigma = -\frac{m\Lambda U}{\pi^2}. \quad (12.16)$$

Then we substitute it back, to find

$$\Sigma = -\frac{m\Lambda U}{\pi^2} - i\frac{m^{\frac{3}{2}}U\sqrt{E + \frac{m\Lambda U}{\pi^2}}}{\pi\sqrt{2}}. \quad (12.17)$$

In order for this to be true, we assume that

$$E_R \equiv E + \frac{m\Lambda U}{\pi^2} \gg \frac{m^{\frac{3}{2}}U\sqrt{E_R}}{\pi\sqrt{2}}. \quad (12.18)$$

In other words, imaginary part of Σ is much less than the real part. Substituting this back into the Green's function, we find

$$G = \frac{1}{E_R - \frac{p^2}{2m} + i\frac{m^{\frac{3}{2}}U\sqrt{E_R}}{\pi\sqrt{2}}}. \quad (12.19)$$

The pole of this Green's function is approximately at

$$E_R = \frac{p^2}{2m} - i\frac{m^{\frac{3}{2}}U\sqrt{p^2/(2m)}}{\pi\sqrt{2}}. \quad (12.20)$$

Thus the particle has a finite lifetime. This lifetime should be interpreted as the time between collisions with the random potential, τ. Then we can write

$$G = \frac{1}{E_R - \frac{p^2}{2m} + i\frac{1}{2\tau}}. \quad (12.21)$$

Note for future reference that

$$\frac{1}{2\tau} = \frac{m^{\frac{3}{2}}U\sqrt{E_R}}{\pi\sqrt{2}} \rightarrow \tau = \frac{1}{2\pi U\nu}, \quad (12.22)$$

where ν is the density of states defined as

$$\nu = \int \frac{d^3p}{(2\pi)^3} \delta\left(E_R - \frac{p^2}{2m}\right). \tag{12.23}$$

It is instructive to recalculate that in the coordinate space

$$G(E,x) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ipx}}{E_R - \frac{p^2}{2m} + i\frac{1}{2\tau}} = \int \frac{pdp}{2\pi^2} \frac{e^{ipx} - e^{-ipx}}{ix\left(E_R - \frac{p^2}{2m} + i\frac{1}{2\tau}\right)} = \frac{m}{\pi x} e^{ix\sqrt{2mE_R}} e^{-|x|\frac{\sqrt{2mE_R}}{4\tau E_R}}. \tag{12.24}$$

This function decays as $e^{-|x|/\ell}$ with the characteristic length

$$\ell = \frac{4\tau E_R}{\sqrt{2mE_R}} = 4\tau v, \tag{12.25}$$

where $v = E_R/\sqrt{2mE_R}$ is the velocity of the particle. In other words, this is the typical length a particle travels between collisions with the random potential (mean free path).

On the other hand, the imaginary part of this Green's function is

$$\text{Im } G(E,x) = \text{Im } \sum_n \frac{\psi_n(x_f)\psi_n^*(x_i)}{E - E_n + i0} = -\pi \sum_n \psi(x_f)\psi^*(x_i)\delta(E - E_n) \tag{12.26}$$

has the meaning of how the wave functions decay at separations $x = x_f - x_i$. The reason this falls off as the mean free path lies in the averaging over random potential: different random potentials introduce different phase in the product $\psi(x_f)\psi^*(x_i)$ and averaging over these potentials makes this product decay with distance.

When deriving everything above, we assumed that $E_R\tau \gg 1$. This condition is also the one which allows one to use the self-consistent Born approximation in the first place.

Indeed, the self-energy diagram shown on Fig. 1 contributes $\sim E + iU\sqrt{E}$, where the term proportional to E comes from the divergent piece of the integral (actually it is $U\Lambda$ which is of the order of E itself). The small parameter $1/(E_R\tau)$ is roughly $U/\sqrt{E} \ll 1$. The second diagram on Fig. 1 (the "rainbow" diagram we actually took into account) corresponds to the integral

$$\int \frac{d^3pd^3q}{(2\pi)^6} \frac{1}{(E - \frac{p^2}{2m} + i0)(E - \frac{q^2}{2m} + i0)^2} \sim \left(E + iU\sqrt{E}\right) \frac{iU}{\sqrt{E}} \sim iU\sqrt{E}, \tag{12.27}$$

that is, it is at the same order as the first rainbow diagram. While the first diagram from Fig. 1, the one which was neglected, is

$$\int \frac{d^3pd^3q}{(2\pi)^3} \frac{1}{\left(E - \frac{p^2}{2m} + i0\right)\left(E - \frac{q^2}{2m}\right)\left(E - \frac{(\vec{p}+\vec{q})^2}{2m} + i0\right)}. \tag{12.28}$$

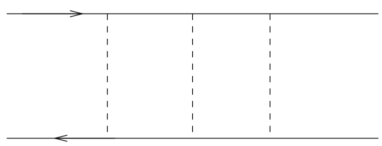

Figure 3: The ladder diagrams which need to be summed to compute (12.29).

The imaginary part of this expression, as is easy to check using Im $1/(x+i0) = -\pi\delta(x)$, is given by a convergent integral, and must thus have the form of U^2/E. This is much smaller than the rainbow diagram contribution, and thus should be neglected.

12.4 Diffusion

To see how a particle moves in a random potential, we need to study the probability that a particle which was at a point x_i at a time t_i ends up at a point x_f at a time t_f. This is given by

$$P(x_f, x_i; t_f, t_i) = \left| G^R(x_f, x_i; t_f, t_i) \right|^2 = G^R(x_f, x_i; t_f, t_i) G^A(x_i, x_f; t_i, t_f). \tag{12.29}$$

In the momentum space, this is equivalent to

$$P(E, p) = \int \frac{d^3q d\omega}{(2\pi)^3} G^R\left(\omega + \frac{E}{2}, q + \frac{p}{2}\right) G^A\left(\omega - \frac{E}{2}, q - \frac{p}{2}\right). \tag{12.30}$$

We need to average this over the random potential. In other words, we need to learn how to compute the average

$$\left\langle G^R\left(\omega + \frac{E}{2}, q + \frac{p}{2}\right) G^A\left(\omega - \frac{E}{2}, q - \frac{p}{2}\right)\right\rangle. \tag{12.31}$$

We compute this average by doing the usual diagrams. Some disorder lines begin and end on the same Green's functions. They change the Green's functions into the average Green's functions computed in the previous subsection. Some connect two Green's functions. Among them, the most important are the ladder diagrams, such as shown on Fig. 3. The following quantity is useful

$$Y(E, \omega, p) \equiv \int \frac{d^3q}{(2\pi)^3} G^R\left(\omega + \frac{E}{2}, q + \frac{p}{2}\right) G^A\left(\omega - \frac{E}{2}, q - \frac{p}{2}\right). \tag{12.32}$$

Its integral over ω gives (12.30). It is given by, in the ladder approximation,

$$Y = \frac{1}{\Pi^{-1} - U}, \tag{12.33}$$

where

$$\Pi(E,\omega,p) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\omega + \frac{E}{2} - \frac{(\vec{p}/2+\vec{q})^2}{2m} + \frac{i}{2\tau}} \frac{1}{\omega - \frac{E}{2} - \frac{(\vec{p}/2-\vec{q})^2}{2m} - \frac{i}{2\tau}}. \tag{12.34}$$

In general, this is a difficult integral to compute. We compute it only at small p, small E, small $1/\tau$, by approximating $q^2/(2m) = q_F^2/(2m) + \xi$, where $q_F = \sqrt{2m\omega}$, and expanding everything up to terms linear in ξ. This gives

$$\frac{1}{2}\int_{-1}^{1} du \int_{-\infty}^{\infty} \nu d\xi \frac{1}{\left(\frac{E}{2} - \frac{pv_Fu}{2} - \xi + \frac{i}{2\tau}\right)\left(-\frac{E}{2} + \frac{pv_Fu}{2} - \xi - \frac{i}{2\tau}\right)}. \tag{12.35}$$

Here $\nu = mq_F/(2\pi^2)$ is the density of states defined by

$$\nu = \int \frac{d^3q}{(2\pi)^3} \delta\left(\omega - \frac{q^2}{2m}\right), \tag{12.36}$$

$v_F = q_F/m$ and the integral over ξ can be extended to infinity due to its convergence. Doing the integral over ξ gives

$$\Pi(E,\omega,p) = -\pi i\nu \int_{-1}^{1} du \frac{1}{-E + pv_Fu - \frac{i}{\tau}} = -\frac{\pi i\nu}{v_Fp} \log\left[\frac{E + \frac{i}{\tau} - pv_F}{E + \frac{i}{\tau} + pv_F}\right]. \tag{12.37}$$

The ω-dependence is buried in the ω and v_F dependence of ν. Expanding $1/\Pi$ for small E and p gives

$$\Pi^{-1} \approx \frac{1}{2\pi\nu\tau} + \frac{v_F^2\tau p^2}{6\pi\nu} - i\frac{E}{2\pi\nu}. \tag{12.38}$$

Finally, this gives

$$Y(p,E;\omega) \approx \frac{2\pi\nu}{\frac{v_F^2\tau p^2}{3} - iE}. \tag{12.39}$$

Here (12.22) was used which led to the cancellation of terms independent of E and p in Y. This is nothing but the Fourier transform of the diffusion operator, with the diffusion constant

$$D = \frac{v_F^2\tau}{3}. \tag{12.40}$$

Thus a particle in a random potential undergoes diffusion: something which is almost obvious if we simply used the formalism of a classical brownian particle.