Quantum Many Body Theory

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Week 10

11 Weakly interacting bosons and superfluidity

11.1 Noninteracting Bose gas

The main difference between weekly interacting bosons and fermions is that bosons form a Bose-Einstein condensate. What this means is, a finite fraction of all the bosons are in the lowest momentum state. As a result,

$$\langle N-1|\, \hat{a}_0 \,|N\rangle = \sqrt{N_0}, \tag{11.1}$$

where N_0 is a finite fraction of the total particle number N. We cannot treat $\hat{a}_0$ and $\hat{a}_p$ for $p \neq 0$ on the equal footing, as $\hat{a}_0$ is, in the limit of a large number of particles, very large. The accepted approach is to separate $\hat{a}_0$, $\hat{a}_0^\dagger$ from other creation and annihilation operators. Then the Green's function only describes the $p \neq 0$ modes. For the noninteracting system, all particles are in the condensate, thus the Green's function for the particles at $p \neq 0$ coincides with the Green's function of the vacuum,

$$G_0(p, E) = \frac{1}{E - \frac{p^2}{2m} + \mu + i0}. \tag{11.2}$$

Here μ is arbitrary (although we know that for the noninteracting Bose gas $\mu = 0$, however having in mind the applications to the interacting Bose gas, we keep it arbitrary.

In addition to the usual Green's functions, the so-called “anomalous” Green's functions are also important. These are defined as

$$F(p, t_f - t_i) = -i \langle N-2|\, \hat{a}_p(t_f)\hat{a}_p(t_i) \,|N\rangle\,. \tag{11.3}$$

Analogously, we can define $F^\dagger$, given by

$$F^\dagger = -i \langle N|\, \hat{a}_p^\dagger(t_f)\hat{a}_p^\dagger(t_i) \,|N-2\rangle\,. \tag{11.4}$$

For the noninteracting Bose gas, these are obviously zero, $F_0 = 0$.

11.2 Diagrammatic expansion

Since the expectation value of $\hat{a}_0$ and $\hat{a}_0^\dagger$ are both equal to $\sqrt{N_0}$, we replace these operators by this number. In practice, we do it in the coordinate representation, which amounts to writing

$$\hat{\psi}(x) = \hat{\psi}'(x) + \sqrt{n_0}, \tag{11.5}$$

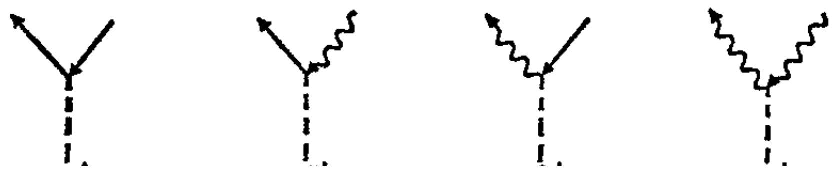

Figure 1: Possible vertices: the dotted line is the interaction, while the wavy lines are the factors of $\sqrt{n_0}$.

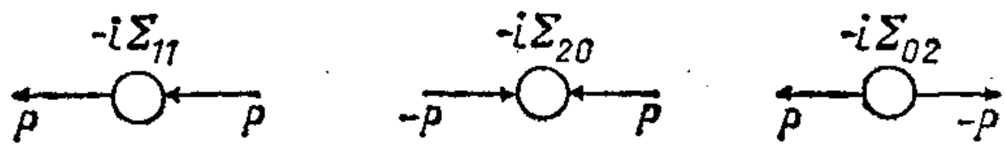

Figure 2: Three types of self energy

where n_0 is the condensate density (which appears because the relationship between $\hat{a}_p$ to $\hat{\psi}(x)$ involves a factor of a square root of volume, $\sqrt{V}$, which converts numbers into density), and $\hat{\psi}'(x)$ is defined by

$$\hat{\psi}'(x) = \frac{1}{\sqrt{V}} \sum_{p \neq 0} \hat{a}_p \, e^{ipx}. \tag{11.6}$$

To do the proper perturbation theory, we need to substitute (11.5) into the interactions, and separate pieces involving different powers of $\sqrt{n_0}$. We denote these terms by drawing the four possible vertices such as shown on Fig. 1. Then each diagram is characterized by the bosonic Green's functions, interactions, and the "condensate lines". These condensate lines do not carry any momentum or energy, but each corresponds to a factor of $\sqrt{n_0}$.

11.3 Self energy diagrams

There are three types of self energies for the Green's function, a shown on Fig. 2. These are denoted by Σ_{11} or Σ_{20} according to the number of lines which enter and exit the diagram. Examples of such diagrams, in the second order, are shown on Fig. 3. It is clear that $\Sigma_{20}(P) = \Sigma_{20}(-P)$ where P is energy and momentum, because P and $-P$ enter its definition symmetrically. It is also clear that $\Sigma_{20}(P) = \Sigma_{02}(P)$,

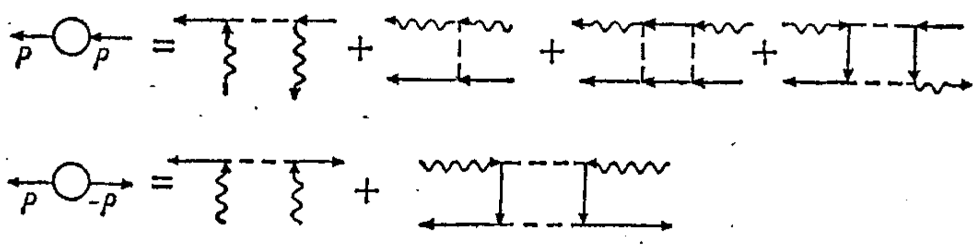

Figure 3: All the second order diagrams which contribute to the normal self energy Σ_{11} (upper row) and the anomalous self energy Σ_{02} (lower row).

11.4 Dyson's equations

Assuming all the self energy diagrams are known, we need to calculate Green's functions. This is not as easy as in the (standard) case of fermions as there are two types of self energy diagrams. We use the following argument: the exact Green's function G consists of G_0, plus all the diagrams which end on Σ_{11} plus all the diagrams which end on Σ_{02}. This allows us to write

$$G(P) = G_0(P) + G(P)\Sigma_{11}G_0(P) + F(P)\Sigma_{02}(P)G_0(P). \tag{11.7}$$

We need a second equation, since the unknowns are $G(P)$ and $F(P)$. It is

$$F(P) = G(-P)\Sigma_{11}(-P)G_0(P) + G(-P)\Sigma_{02}(P)G_0(P). \tag{11.8}$$

The solutions are

$$G(P) = \frac{E + \frac{p^2}{2m} - \mu + \Sigma_{11}(-P)}{\left[\Sigma_{02}(P)\right]^2 - \left[\Sigma_{11}(P) - E - i0 + \frac{p^2}{2m} - \mu\right]\left[\Sigma_{11}(-P) + E + i0 + \frac{p^2}{2m} - \mu\right]}, \tag{11.9}$$

$$F(P) = -\frac{\Sigma_{02}(P)}{\left[\Sigma_{02}(P)\right]^2 - \left[\Sigma_{11}(P) - E - i0 + \frac{p^2}{2m} - \mu\right]\left[\Sigma_{11}(-P) + E + i0 + \frac{p^2}{2m} - \mu\right]}. \tag{11.10}$$

The low-energy excitations of such a system are phonons (Bogoliubov's modes) and their energy vanishes with their momentum. Thus at $p = E = 0$, the denominators of (11.9) and (11.10) vanishes (so that the Green's function has a pole at $E = 0$, $p = 0$). This gives a condition on μ:

$$\left(\Sigma_{11}(0) - \mu\right)^2 = \left(\Sigma_{02}(0)\right)^2. \tag{11.11}$$

This gives

$$\mu = \Sigma_{11}(0) - \Sigma_{02}(0). \tag{11.12}$$

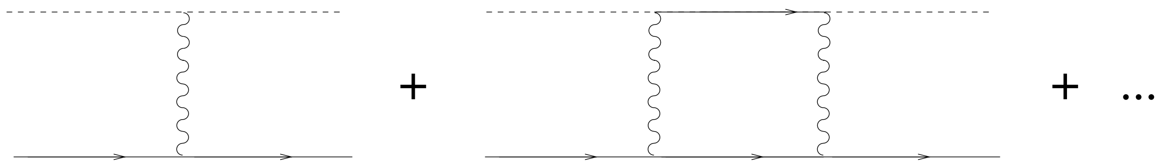

Figure 4: All the diagrams which contribute to the normal self energy (dotted lines are those of the condensate, while wavy line is that of the interactions, opposite of the previous figures in the notes)

This is called the Pines-Hugenholtz relation. Its importance is in the following: unlike the usual diagrammatic formalism, here we have a parameter, n_0, which enters the expressions for the diagrams, and which we do not know in advance. This relationship, which effectively relates n_0 and μ, fixes this unknown.

11.5 Solution to the problem of the low density interacting Bose gas

Suppose a Bose gas consists of bosons which interact repulsively via some two-body interaction, such that its scattering a is sufficiently small, or

$$na^3 \ll 1, \tag{11.13}$$

where n is the density. In this case, the interaction can be strong, so expanding in powers of interaction is not a good approximation. However, expanding in powers of density n, or equivalently in powers of the condensate density $n_0 < n$, is legitimate. The lowest diagrams are those where there are two condensate lines. Such diagrams are shown on Fig. 4. All the Green's functions happen to coincide with those in vacuum, and external momentum is set to zero, so the sum of all such diagrams is simply $(4\pi/m)a$ or the scattering amplitude at zero momentum in vacuum. There is also an overall factor of 2 due to the fact that either of the two outgoing lines can be the condensate line (either of the two incoming lines can be the condensate line as well, but that is already taken into account by canceling the factor of 1/2 in the interactions). This gives

$$\Sigma_{11}(0) = \frac{8\pi}{m} a n_0. \tag{11.14}$$

There are also anomalous self-energy diagrams which are basically the same as the normal diagrams but with external lines exchanged. These diagrams, at zero energy-momentum,

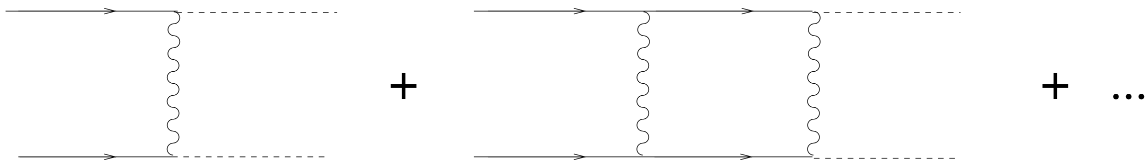

Figure 5: All the diagrams which contribute to the anomalous self energy (dotted lines are those of the condensate, while wavy line is that of the interactions, opposite of the previous figures in the notes

also sum into the scattering amplitude in vacuum (except no factor of 2). This gives

$$\Sigma_{20}(0) = \frac{4\pi}{m} a n_0. \tag{11.15}$$

This leads to, by virtue of (11.12),

$$\mu = \frac{4\pi}{m} a n_0. \tag{11.16}$$

This also allows us to compute the Green's function (11.9) (at low enough energy and momentum, so that Σ can still be approximated by its zero energy-momentum value, which implies $p \ll 1/a$, $E \ll 1/(ma^2)$)

$$G(P) = \frac{E + \frac{p^2}{2m} + \frac{4\pi a n_0}{m}}{\left[\frac{4\pi a n_0}{m}\right]^2 - \left[\frac{4\pi a n_0}{m} - E - i0 + \frac{p^2}{2m}\right]\left[\frac{4\pi a n_0}{m} + E + i0 + \frac{p^2}{2m}\right]}, \tag{11.17}$$

The poles of the denominator occur at

$$E = \pm\sqrt{\left(\frac{p^2}{2m}\right)^2 + \frac{4\pi a n_0 p^2}{m^2}}. \tag{11.18}$$

This is the spectrum of the Bogoliubov excitations in a condensate. The number of particles can be calculated by, as usual,

$$\lim_{t\to -0} \int \frac{d^3p dE}{(2\pi)^4} e^{-iEt} \frac{E + \frac{p^2}{2m} + \frac{4\pi a n_0}{m}}{\left[\frac{4\pi a n_0}{m}\right]^2 - \left[\frac{4\pi a n_0}{m} - E - i0 + \frac{p^2}{2m}\right]\left[\frac{4\pi a n_0}{m} + E + i0 + \frac{p^2}{2m}\right]} = \frac{8 n_0}{3}\sqrt{\frac{n_0 a^3}{\pi}}. \tag{11.19}$$

This is the number of particles not in the condensate, that is, this is $n - n_0$. Thus

$$n - n_0 = \frac{8 n_0}{3}\sqrt{\frac{n_0 a^3}{\pi}} \approx \frac{8 n}{3}\sqrt{\frac{n a^3}{\pi}}, \tag{11.20}$$

where in the last approximate equality we notice that this number is small so within this order in na^3 one can replace n by n_0. This gives the condensate density n_0 in terms of the total density n. Notice that since $n_0 < n$, not all the particles are Bose condensed, even though we work at zero temperature. The interactions "expel" some particles from the condensate.