Problem Set 6

Phys 7280
Due: May 1

1 Cooper's problem

Before the theory of superconductivity was invented, L. Cooper solved the following problem. Consider two attractively interacting fermions with opposite spin. Imagine that they move in the presence of a gas of fermions filling a Fermi sphere. However, neglect the interactions between the two fermions and the fermions filling the Fermi sphere. Thus the role of the fermions filling the sphere is simply to restrict the momenta of the interacting fermions to lie above the Fermi momentum. Such fermions, as L. Cooper showed, can form a bound state even when the interactions are arbitrarily weak.

1. First let us solve the problem of two fermions interacting via an attractive interaction in vacuum. The Schrödinger equation for the two fermions is given by

$$E\Psi(\mathbf{r},\mathbf{r}') = \left(-\frac{\Delta_{\mathbf{r}}}{2m} - \frac{\Delta_{\mathbf{r}'}}{2m}\right)\Psi(\mathbf{r},\mathbf{r}') - \lambda\delta(\mathbf{r}-\mathbf{r}')\Psi(\mathbf{r},\mathbf{r}'). \tag{1.1}$$

To solve this equation, we Fourier transform it to go to the momentum space

$$E\Psi(\mathbf{p},\mathbf{p}') = \left(\frac{p^2}{2m} + \frac{p'^2}{2m}\right)\Psi(\mathbf{p},\mathbf{p}') - \frac{\lambda}{V}\sum_{\mathbf{q},\mathbf{q}'}\delta_{\mathbf{p}+\mathbf{p}',\mathbf{q}+\mathbf{q}'}\Psi(\mathbf{q},\mathbf{q}'). \tag{1.2}$$

Here $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{q}$, $\mathbf{q}'$ are all restricted to lie below the inverse range of the potential, or $p < \Lambda$, $p' < \Lambda$, $q < \Lambda$, and $q' < \Lambda$. Then we go to the center-of-mass reference frame, which implies $\mathbf{p} = -\mathbf{p}'$, $\mathbf{q} = -\mathbf{q}'$, to give

$$E\Psi(\mathbf{p}) = \frac{p^2}{m}\Psi(\mathbf{p}) - \frac{\lambda}{V}\sum_{q<\Lambda}\Psi(\mathbf{q}), \tag{1.3}$$

where we introduce the notation $\Psi(\mathbf{p}) \equiv \Psi(\mathbf{p},-\mathbf{p})$.

Solve the equation (1.3) for $E < 0$. Show that there is a solution to this equation (corresponding to a bound state at energy $E < 0$) only if $\lambda > \lambda_c$. Find λ_c in terms of Λ and m. (Convert the summation over q to the integration over q to do that).

2. Now the Cooper's problem. The two fermions move in the presence of a Fermi sea, so the Schrödinger equation now reads

$$E\Psi(\mathbf{p}) = \left(\frac{p^2}{m} - 2\mu\right)\Psi(\mathbf{p}) - \frac{\lambda}{V}\sum_{q_F<q<\Lambda}\Psi(\mathbf{q}), \tag{1.4}$$

where $q_F^2/(2m) = \mu$. Solve this equation for $E < 0$. Show that the solution exist no matter how small λ is.

2 Bogoliubov-de-Gennes equations

Consider a superconductor in the presence of a gap function Δ which depends on the position in space,

$$\hat{\tilde{H}} = \sum_{p,\alpha=\uparrow,\downarrow} \left(\frac{p^2}{2m} - \mu \right) \hat{a}^\dagger_{\alpha,p} a_{\alpha,p} + \sum_p \Delta_p \sum_q \hat{a}^\dagger_{\uparrow,\frac{p}{2}+q} \hat{a}^\dagger_{\downarrow,\frac{p}{2}-q} + \sum_p \Delta^*_p \sum_q \hat{a}_{\downarrow,\frac{p}{2}-q} \hat{a}_{\uparrow,\frac{p}{2}+q} + \frac{V}{\lambda} \sum_p \Delta^*_p \Delta_p. \tag{2.1}$$

In the coordinate space, this Hamiltonian can be written as

$$\hat{\tilde{H}} = \int d^3x \left[\sum_{\alpha=\uparrow,\downarrow} \hat{\psi}^\dagger_\alpha(x) \left(-\frac{\Delta_{\mathbf{r}}}{2m} - \mu \right) \hat{\psi}_\alpha(x) + \Delta(x) \hat{\psi}^\dagger_\uparrow(x) \hat{\psi}^\dagger_\downarrow(x) + \Delta^*(x) \hat{\psi}_\downarrow(x) \hat{\psi}_\uparrow(x) + \frac{1}{\lambda} \Delta(x) \Delta^*(x) \right]. \tag{2.2}$$

Here, for lack of better notations, $\Delta_{\mathbf{r}}$ is the Laplace operator, while $\Delta(x)$ or Δ_p is the gap function.

1. Exercise in Fourier transform. Find the correct relationship between Δ_p and $\Delta(x)$ (with the correct factors of volume V) so that (2.1) and (2.2) are consistent with each other. Recall that $\hat{a}$, $\hat{a}^\dagger$, $\hat{\psi}$, and $\hat{\psi}^\dagger$ are related by (3.8), (3.9) of Week 3 notes (where L stands for volume V).

2. Find the Dyson's equations which G and F, defined in (15.15) of Week 15 notes satisfy. These will be the appropriate generalizations of (15.17) of the notes. Try to find them first in the momentum space, and then think about how to write them in the coordinate space. These equations in the coordinate space are called the Bogoliubov-de-Gennes equations. They are used, for example, to find the spectrum of the Bogoliubov excitations in the presence of a vortex, or when $\Delta(x) = \Delta_0 e^{i\phi}$ where ϕ is the polar angle and Δ_0 is some position independent constant. More can be found in C. Caroli, P.G. de Gennes, J. Matricon, Phys. Lett. **9**, 307 (1964) (available online, at http://www.sciencedirect.com/science/journal/00319163).