Problem Set 5

Phys 7280
Due: Apr 14

1 Level correlations in random matrix ensembles

Consider a random Gaussian Hermitian matrix H_{ij}, such that

$$\langle H_{ij} H_{kl} \rangle = h \delta_{il} \delta_{jk}, \tag{1.1}$$

just as discussed in class. Suppose E_n are its eigenvalues. The density of states is defined by

$$\rho(E) = \frac{1}{N} \sum_n \delta(E - E_n). \tag{1.2}$$

In class we calculated

$$\langle \rho(E) \rangle = \frac{\sqrt{4hN - E^2}}{2\pi h N}, \tag{1.3}$$

where brackets denote averaging over random H_{ij}.

Now we would like to calculate the density-density correlation function

$$R(E; \epsilon) = \langle \rho(E + \epsilon) \rho(E) \rangle - \langle \rho(E) \rangle \langle \rho(E + \epsilon) \rangle. \tag{1.4}$$

To do that, we use the relation

$$\rho(E) = \frac{i}{2\pi N} \sum_{i=1}^{N} \left(G^R_{ii}(E) - G^A_{ii}(E) \right), \ G^R = \left[E - H + i0 \right]^{-1}. \tag{1.5}$$

1. Substitute (1.5) into (1.4) and express the density density correlation functions in terms of expressions $\left\langle G^R(E) G^R(E+\epsilon) \right\rangle$, $\left\langle G^R(E) G^R(E+\epsilon) \right\rangle$, and $\left\langle G^A(E) G^R(E+\epsilon) \right\rangle$, $\left\langle G^R(E) G^A(E+\epsilon) \right\rangle$.
2. Discuss the diagrams which will contribute to these averages in the large N approximation (these will be a suitable adaptation of the ladder diagrams introduced in class).
3. Sum these diagrams and calculate $R(E; \epsilon)$ in the limit when $E \ll 4hN$, $\epsilon \ll 4hN$. (Don't be surprised if the correlation function diverges at very small ϵ. The ladder diagram trick breaks down when ϵ is too small.)

2 The Lloyd model

In 1969 P. Lloyd solved the following problem exactly. Consider the following discrete Schrödinger equation

$$2\psi_k - \psi_{k-1} - \psi_{k+1} + V_k\psi_k = E\psi_k, \tag{2.1}$$

where integer index $k = 0, 1, \ldots, N-1$ labels points on the lattice, and we assume periodic boundary conditions, $\psi_N = \psi_0$. (This of course approximates the usual continuous 1D Schrödinger equation, since the first three terms look like the lattice second derivative, and the fourth term looks like a position-dependent potential). Now suppose V_k, for each k, are independent random variables governed by the following probability distribution

$$P(V) = \frac{\gamma}{\pi\left(V^2+\gamma^2\right)}, \tag{2.2}$$

where γ is a parameter (disorder strength). Following Lloyd, you will find the average density of states for this problem exactly. The density of states is defined by

$$\rho(E) = \frac{1}{N}\sum_n \delta(E-E_n) = -\frac{1}{N\pi}\mathrm{Im}\,\sum_{i=1}^{N} G^R_{ii}(E). \tag{2.3}$$

1. To calculate the retarded Green's function, we make use of the following (Schwinger) representation

$$G(E) = \left[E + i0 - \hat{H}_0 - \hat{V}\right]^{-1} = i\int_0^\infty ds\ e^{is\left(E+i0-\hat{H}_0-\hat{V}\right)}. \tag{2.4}$$

Here $\hat{H}_0$ and $\hat{V}$ are matrices such that $\left(\hat{H}_0+\hat{V}\right)\psi$ reproduces the Schrd̈inger equation (2.1), that is, $\hat{H}_{0kl} = 2\delta_{k,l} - \delta_{k,l-1} - \delta_{k,l+1}$ and $\hat{V}_{kl} = V_k\delta_{kl}$. Now we can average over random potential V_k, by making use of the formula

$$\int_{-\infty}^{\infty} dV\ e^{isV}\frac{\gamma}{\pi\left(V^2+\gamma^2\right)} = e^{-s\gamma} \tag{2.5}$$

true if $s > 0$. Use it to prove that the average retarded Green's function is

$$\left\langle G^R(E)\right\rangle = G^R_0(E+i\gamma),\ G^R_0 = \left[E + i0 - \hat{H}_0\right]^{-1}. \tag{2.6}$$

2. Calculate $G^R_0(E+i\gamma)$ and use it to compute the density of states (2.3).
3. Discuss why this technique breaks down if we would want to calculate $\left\langle G^R(E_1)\,G^A(E_2)\right\rangle$ (as a result, the diffusion and localization cannot be studied using this trick).