Problem Set 4

Phys 7280
Due: Apr 2

1 Plasma oscilations

Consider an electron gas with density n and chemical potential μ which interacts via Coulomb potential $V(r) = e^2/r$ where r is the distance between the electrons. Its Fourier transform is given by $V(q) = 4\pi e^2/q^2$. An important parameter for this gas is the ratio of its typical Coulomb and kinetic energies, denoted $r_0 = e^2 n^{\frac{1}{3}}/\mu$. In what follows, we assume that $r_0 \ll 1$ so that the interactions can be considered weak and perturbation theory in powers of interaction can be used. As discussed, the interaction potential is modified by the RPA diagrams as shown on Fig. 1. The basic building block of these diagrams is the polarization operator which plays the role of the "self-energy" correction to the Coulomb interactions.

The sum of these diagrams represents an approximation to the vertex function, as is emphasized by the four external lines on each of the diagrams shown on Fig. 1. In turn, that function can be thought of as describing the electron-hole excitations in the gas (plasmons), and the poles in that function give their dispersion relation.

1. Use the explicit form of the polarization operator $\Pi(E, p)$ found in Problem 3 of the Problem Set 3 to sum all the diagrams shown on Fig. 1.

2. By analyzing the behavior of the polarization operator at $p \ll p_F$, $E \ll \mu$, find the spectrum of the plasmons $\epsilon(p)$ at low momentum. *Hint:* Assume that $E \gg p p_F/m$.

3. There exists such $p_{\max}$ that the plasmon spectrum cannot be extended for larger $p > p_{\max}$, at least if E is real. Find this $p_{\max}$.

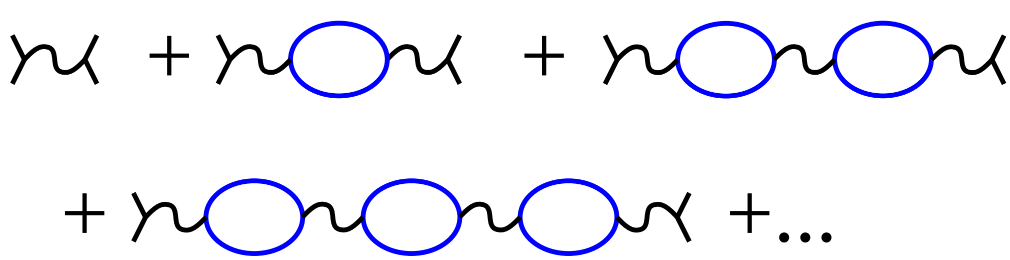

Figure 1: The RPA diagrams which describe the screening of the Coulomb potential by the electron gas.

2 Interaction screening

Summing the diagrams in the previous problem, one finds the renormalized interaction $V_{\text{ren}}(E, p)$. By calculating the Fourier transform of $V_{\text{ren}}(0, p)$ with respect to p, find the effective (screened) interaction between the electrons in the gas. *Hint:* In doing the Fourier transform, simplify the problem by assuming that only small $p \ll p_F$ contribute and thus the polarization operator can be simplified as in the part 1 of the previous problem.

3 Composite bosons

In the experiments done in JILA, Bose-Einstein condensation of composite bosons is studied. These bosons can decay into fermions of opposite spin, but otherwise do not interact. The Green's function of the fermions is given by

$$G_0(E, p) = \frac{1}{E - \frac{p^2}{2m} + \mu + i0 \,\text{sign}\,(E)}, \tag{3.1}$$

and the Green's function of the bosons is given by

$$D_0(E, p) = \frac{1}{E - \epsilon - \frac{p^2}{4m} + 2\mu + i0}. \tag{3.2}$$

Here $\epsilon < 0$ is the binding energy (the energy released when fermions combine together to form bosons). 2μ in the Green's function signifies the fact that it is *two* fermions which combine together to form a boson. The interaction term is given by

$$\hat{H}_{\text{int}} = g \int d^3x \left[\hat{\phi}\, \hat{\psi}_\uparrow^\dagger \psi_\downarrow^\dagger + \hat{\phi}^\dagger\, \hat{\psi}_\downarrow \hat{\psi}_\uparrow \right]. \tag{3.3}$$

We will try to solve this problem assuming g is small in the perturbative expansion in powers of g. Notice that the interaction term, as well as D, do not have anything to do with the corresponding expressions in Problem 1 of the Problem Set 3 (a common mistake is to confuse them).

1. Identify the lowest order bosonic self energy diagram (proportional to g^2). There is only normal self energy at this order in g, and no anomalous self energy. This diagram corrects D_0. Calculate its value at zero energy and momentum, assuming that the momentum to be integrated over cannot exceed a maximum momentum p_{max} (just like we did in the problem 1 of the problem set 3). Show that this value, which we denote $\Sigma_{11}^{(1)}(0, 0)$, gets added to ϵ. In what follows, we assume that $\Sigma_{11}^{(1)}(0, 0) + \epsilon < 0$.

2. Identify the next self energy diagrams, proportional to g^4, which we can denote by $\Sigma_{11}^{(2)}$ and $\Sigma_{02}^{(2)}$. These diagrams, both normal and anomalous, are also proportional to n_0, the condensate density. Compute the value of these diagrams at zero external energy and momentum.

3. Find μ in terms of n_0 using the Pines-Hughenholtz relation.

4. We expect that n_0 is different from n by terms small if g is small. Replacing n_0 by $n = N/V$, find the ground state energy E_0 of the gas (remember, $dE_0/dN = \mu$).