Problem Set 1

Phys 7280
Due: Feb 11

1 Green's function of a free particle

Consider a particle moving on an infinitely long line. The Hamiltonian is given by, as usual

$$\hat{H} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}. \tag{1.1}$$

In class, we calculated its Green's function

$$G_0(x, E) = -\sqrt{\frac{m}{-2E}}\, e^{-\sqrt{-2mE}\,|x|},\ x = x_f - x_i. \tag{1.2}$$

However, this is one of four different forms of the Green's functions for a free particle. There are three other forms, $G_0(k, E)$, $G_0(x, t)$, $G_0(k, t)$. For example, in class we also showed that

$$G_0(k, E) = \frac{1}{E - \frac{k^2}{2m} + i0}. \tag{1.3}$$

Calculate $G_0(k, t)$ and $G_0(x, t)$.

2 Harmonic Oscillator by Cteation and Annihilation Operators

A Hamiltonian for a harmonic oscillator is given by

$$\hat{H} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\,\omega^2 x^2}{2}. \tag{2.1}$$

Although one can find the energy levels of this hamiltonian by solving the appropriate Schr"odinger equation, there exists a shortcut.

We define an operator

$$\hat{a} = \frac{1}{\sqrt{2}}\left[\sqrt{m\omega}\,x + \frac{1}{\sqrt{m\omega}}\frac{\partial}{\partial x}\right]. \tag{2.2}$$

1. Calculate $\hat{a}^\dagger$.

2. Show that $\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$.

3. Show that

$$\hat{H} = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \tag{2.3}$$

4. Find the energy levels of the oscillator.

5. The ground state of the oscillator satisfies $\hat{a}\,|0\rangle = 0$. Use that to find the wave function of the ground state.

3 Bogoliubov tranformation

Consider a problem where bosons may accumulate on a site. The energy of a boson on a site is ϵ, while the nature of the problem is such that the bosons appear or disappear in groups of 2, with the probability amplitude t. The Hamiltonian is given by

$$\hat{H} = \epsilon\, \hat{a}^\dagger \hat{a} + t \left(\hat{a}^2 + \hat{a}^{\dagger\,2} \right). \tag{3.1}$$

Here $\hat{a}$, $\hat{a}^\dagger$ are the annihiation and creation operator for these bosons. Notice that the total number of bosons in this problem is not conserved.

We will employ the Bogoliubov transformation to solve the problem It is defined in the following way

$$\hat{b} = \hat{a} \cosh(\phi) + \hat{a}^\dagger \sinh(\phi),\ \ \hat{b}^\dagger = \hat{a}^\dagger \cosh(\phi) + \hat{a} \sinh(\phi). \tag{3.2}$$

Correspondingly,

$$\hat{a} = \hat{b} \cosh(\phi) - \hat{b}^\dagger \sinh(\phi),\ \ \hat{a}^\dagger = \hat{b}^\dagger \cosh(\phi) - \hat{b} \sinh(\phi). \tag{3.3}$$

1. Check that $\hat{b}\,\hat{b}^\dagger - \hat{b}^\dagger \hat{b} = 1$ for any ϕ. In other words, these are genuine creation and annihilation operators.
2. Express a in terms of b, and substitute into the Hamiltonian (3.1). Find such ϕ that the Hamiltonian can be reduced to the form

$$\hat{H} = \omega\, \hat{b}^\dagger \hat{b} + \Omega. \tag{3.4}$$

Find ω and Ω, in terms of ϵ and t.
3. The ground state of the system corresponds to the absence of particles b. But that doesn't mean there are no a-particles there. Indeed, in the ground state $\left\langle \hat{b}\,\hat{b}^\dagger \right\rangle = 1$ (while of course $\left\langle \hat{b}^\dagger\, \hat{b} \right\rangle = 0$, $\left\langle \hat{b}\,\hat{b} \right\rangle = 0$) and

$$\left\langle \hat{a}^\dagger \hat{a} \right\rangle = \sinh^2 \phi \tag{3.5}$$

Use the exression for ϕ in terms of t and ϵ to find the number of particles in the ground state.