Advanced Statistical Mechanics

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Week 9

22 Quantum Hall Effect

Hall effect is the appearance of a transverse voltage in a metal bar with a current and a magnetic field. In a magnetic field, the Lorentz force acting on the electron has to be compensated by the transverse electric field

$$\frac{e}{c}vB = e\frac{U}{d} \tag{22.1}$$

From here the current can be evaluated

$$I = en_c v = \frac{eUn_c c}{Bd} = \frac{enc}{B}U, \tag{22.2}$$

where n_c is the density per unit length and n is the electron density per unit area. enc/B is called the Hall conductance.

In 2D the electrons are governed by the 2D Schrödinger equation in a magnetic field. We choose the vector potential $A_x = -\frac{1}{2}By$, $A_y = \frac{1}{2}Bx$, and write down the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left[\left(\partial_x + \frac{ie}{2\hbar c}By\right)^2 + \left(\partial_y - \frac{ie}{2\hbar c}Bx\right)^2\right] \tag{22.3}$$

The common notations are $\omega = eB/mc$ is the Larmor frequency. $l = \sqrt{\hbar c/eB}$ is the magnetic length. It is convenient to rewrite the Hamiltonian in terms of dimensionless variables $X = x/\sqrt{2}l$, $Y = y/\sqrt{2}l$, to find

$$H = -\frac{\hbar\omega}{4}\left[(\partial_X + iY)^2 + (\partial_Y - iX)^2\right]. \tag{22.4}$$

We now introduce complex coordinates $z = x + iy$ and magnetic creation and annihilation operators

$$a = \sqrt{2}l\bar{\partial} + \frac{1}{2\sqrt{2}l}z,\ a^\dagger = -\sqrt{2}l\partial + \frac{1}{2\sqrt{2}l}\bar{z},\ \left[a, a^\dagger\right] = 1, \tag{22.5}$$

and express the Hamiltonian in their terms

$$H = \frac{\hbar\omega}{2}\left(a^\dagger a + aa^\dagger\right). \tag{22.6}$$

Also important are magnetic translation operators

$$b = \sqrt{2}l\bar{\partial} - \frac{1}{2\sqrt{2}l}z,\ b^\dagger = \sqrt{2}l\partial + \frac{1}{2\sqrt{2}l}\bar{z},\ \left[b, b^\dagger\right] = 1 \tag{22.7}$$

They commute with a, $a^\dagger$, and therefore, commute with the Hamiltonian.

The energy levels are those of an oscillator $E_n = \hbar\omega\left(\frac{1}{2}+n\right),$. They are called Landau levels. The lowest Landau level wave functions can be found by

$$a\psi_m = 0 \ \rightarrow \psi_m \propto z^m e^{-\frac{z\bar{z}}{4l^2}}. \tag{22.8}$$

They are labelled by the integer m. The m-th wavefunction in the ground state has a spacial extend governed by

$$|\psi_m|^2 = \exp\left[-\frac{z\bar{z}}{2l^2} + m\log\left(z\bar{z}\right)\right] \tag{22.9}$$

or $r^2_{\max} = 2ml^2$. The area covered by the first m states is $2\pi ml^2$. In other words, the density of states in the lowest Landau level is

$$n_{LL} = \frac{1}{2\pi l^2}. \tag{22.10}$$

Introduce the filling factor ν, the ratio of the number of electrons to m in a geometry of the area A. This gives the Hall conductance

$$G = \frac{enc}{B} = \frac{e^2}{2\pi\hbar}\nu. \tag{22.11}$$

If ν is integer, the Hall conductance is quantized.

If the magnetic field is changed while the number of electrons is kept fixed, $\nu = n2\pi l^2 = 2\pi n\frac{\hbar c}{eB}$. In other words, as we're increasing the field, conductance goes down.

The quantum Hall effect arises when the electrons form incompressible states. This happens either at integer ν or, if interactions are taken into account, at fractional values of ν.

The edge of a quantum Hall droplet forms a Luttinger liquid. To see this, we perform a Laughlin experiment. We pierce the quantum Hall sample with a magnetic field until it produces exactly one flux quantum. This creates an electric field along the edge.

$$2\pi rE = -\frac{1}{c}\frac{\partial\Phi}{dt}. \tag{22.12}$$

The flux quantum ($\Phi = 2\pi\hbar c/e$) is reached when

$$\int dt\ E(t)2\pi r = \frac{\hbar}{e}2\pi \tag{22.13}$$

Then the charge transferred to the edge is

$$Q = \int dt\ I(t) = \int dt\frac{e^2}{2\pi\hbar}\nu E2\pi r = \int dt\frac{e^2}{2\pi\hbar}\nu\frac{2\pi\hbar}{e} = \nu e. \tag{22.14}$$

In other words, the current at the edge increases by this amount.

Imagine that a Luttinger liquid g with only right moving particles lives at the edge. The axial anomaly relation will read

$$\delta I = \frac{ge}{2\pi} \int dtdx \; E(t). \tag{22.15}$$

This takes into account that the anomaly is only half as big - no left moving particles to remove. Then we find

$$\delta I = ge \tag{22.16}$$

So, $g = \nu$.