# Advanced Statistical Mechanics

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Week 8

# 21 Tunneling into a Luttinger Liquid

(This is taken mostly from a review article by M. P. A. Fisher and L. I. Glazman.) Tunneling density of states is defined as

$$\rho(E) = 2\pi \sum_n \left|\left\langle n \left|\psi^\dagger(x)\right| 0\right\rangle\right|^2 \delta\left(E_n - E_0 - E\right). \tag{21.1}$$

This can be related to the imaginary part of the correlation function

$$\rho(E) = 2\mathrm{Re} \int_0^\infty dt\ e^{i(E+i\epsilon)t} \left\langle \psi(t,x)\psi^\dagger(0,x)\right\rangle. \tag{21.2}$$

This formula can be derived using

$$\begin{aligned}\rho(E) &= 2\mathrm{Re}\sum_n \int_0^\infty dt\ e^{i(E+E_0-E_n+i\epsilon)t} \left|\left\langle n \left|\psi^\dagger(x)\right| 0\right\rangle\right|^2 = \\ &= 2\mathrm{Re}\sum_n \frac{i}{i\epsilon + E + E_0 - E_n} \left|\left\langle n \left|\psi^\dagger(x)\right| 0\right\rangle\right|^2\end{aligned} \tag{21.3}$$

and the standard representation for the delta function

$$\frac{1}{x+i\epsilon} = \frac{1}{x} - i\pi\delta(x). \tag{21.4}$$

The correlation function can now be evaluated if we use

$$\psi(x) = \psi_L(x) e^{-ik_F x} + \psi_R(x) e^{ik_F x} = e^{-ik_F x} e^{-i\varphi - i\theta} + e^{ik_F x} e^{-i\varphi + i\theta}. \tag{21.5}$$

We now evaluate the tunneling density of states. The correlation function can then be computed with the help of the bosonization formula as

$$\left\langle\ e^{-i\varphi(\tau)-i\theta(\tau)} e^{i\varphi(0)+i\theta(0)}\ \right\rangle,\ \text{and}\ \left\langle\ e^{-i\varphi(\tau)+i\theta(\tau)} e^{i\varphi(0)-i\theta(0)}\ \right\rangle. \tag{21.6}$$

We find

$$\left\langle\ e^{-i\varphi(\tau)-i\theta(\tau)} e^{i\varphi(0)+i\theta(0)}\ \right\rangle \propto \exp\left[\langle\, \varphi(\tau)\varphi(0)\,\rangle + \langle\, \theta(\tau)\theta(0)\,\rangle + \langle\, \theta(\tau)\varphi(0)\,\rangle + \langle\, \varphi(\tau)\theta(0)\,\rangle\right]. \tag{21.7}$$

We already know the first two correlation functions.

$$\langle\, \varphi(\tau)\varphi(0)\,\rangle = -\frac{1}{2g}\log|\tau|,\ \ \langle\, \theta(\tau)\theta(0)\,\rangle = -\frac{g}{2}\log|\tau|. \tag{21.8}$$

The other two correlations can be found if we use

$$\partial_\mu \theta = i g \epsilon_{\mu\nu} \partial_\nu \varphi, \tag{21.9}$$

and are purely imaginary, thus giving an additional phase to (21.7). The phases cancel, and the answer is

$$\langle\, \psi_L(\tau)\psi_L \,\rangle = \langle\, \psi_R\psi_R \,\rangle \propto \left(\frac{\tau_0}{|\tau|}\right)^{\alpha}, \; \alpha = \frac{1}{2}\left(g + \frac{1}{g}\right). \tag{21.10}$$

It is crucial that this formula works only for $\tau \gg \tau_0$. More generally, we have to write

$$\langle\, \psi(\tau)\psi(0) \,\rangle = \left(\frac{\tau_0}{\tau_0 + |\tau|}\right)^{\alpha}. \tag{21.11}$$

The time cutoff $\tau_0$ appear because of energy cutoff $\epsilon_F$.

We can now evaluate (21.2). Analytically continue to $\tau = it$, we find

$$\mathrm{Re} \int_0^{\infty} dt \; e^{i(E+i\epsilon)t} \left(\frac{\tau_0}{it + \tau_0}\right)^{\alpha}. \tag{21.12}$$

Deforming the contour and substituting $t = ix$, we find

$$\mathrm{Im} \int_0^{\infty} dx \; e^{-Ex} \frac{1}{(\tau_0 - x)^{\alpha}}. \tag{21.13}$$

where the integration contour goes below the singularity $x = \tau_0$. The singularity does not contribute to the imaginary part of the integral, since if $E = 0$, the integral from $-\infty$ to $+\infty$ has to be zero (no poles in the upper half plane), while the integral from $-\infty$ to 0 is clearly real. Therefore we are left with

$$\mathrm{Im} \int_0^{\infty} dx \; e^{-Ex + i\pi\alpha} \frac{1}{(x - \tau_0 x)^{\alpha}} = \sin(\pi\alpha)\, \Gamma(1-\alpha) E^{\alpha-1} = \frac{\pi}{\Gamma(\alpha)} E^{\alpha-1}. \tag{21.14}$$

Finally the tunneling density of states is

$$\rho(E) = \frac{2\pi}{\Gamma(\alpha)} E^{\alpha-1}. \tag{21.15}$$

Suppose the electrons can tunnel from a lead to the Luttinger liquid, and the lead is kept at voltage $U$ compared to the liquid. Then the Hamiltonian acquires the tunneling term $H = H_0 + \lambda\psi^{\dagger}(x)\psi_0(x) + \psi_0^{\dagger}(x)\psi(x)$, and the current, via the Fermi golden rule, is proportional to

$$I \propto \int_0^{eV} dE \; \rho(E) \propto V^{\alpha} \tag{21.16}$$

This is the nonlinear conductance of the Luttinger liquid.

Tunneling to the edge of the Luttinger liquid is also of interest. At the edge there's no current, consequently $j^1(x=0) \propto \partial_t\theta(x=0) = 0$. So $\theta(x=0)$ is a constant. Additionally, $\partial_t\theta = \partial_x\varphi$, so $\partial_x\varphi(x=0) = 0$. It follows from here that

$$\langle\, \varphi(x,\tau)\varphi(x',\tau')\,\rangle = -\frac{1}{2g}\left[\log\left(\sqrt{(\tau-\tau')^2+(x-x')^2}\right)+\log\left(\sqrt{(\tau-\tau')^2+(x+x')^2}\right)\right]. \tag{21.17}$$

Then the correlation function, computed at the edge, gives

$$\left\langle\, \psi^\dagger(\tau,0)\psi(0,0)\,\right\rangle \propto \frac{1}{|\tau|^{\frac{1}{g}}}, \tag{21.18}$$

and we get the same I-V characteristics (21.16) with $\alpha = 1/g$.