Advanced Statistical Mechanics

Victor Gurarie

Week 7

18 Bosonization of Free Electrons

The imaginary time bosonization looks like

$$S = \frac{1}{2\pi} \int d^2\xi \, (\partial_\mu \theta)^2 \,, \tag{18.1}$$

where the electron current is

$$j_\mu = -\frac{i}{\pi} \epsilon_{\mu\nu} \partial_\nu \theta. \tag{18.2}$$

It is instructive to go back to the real time formalism. The action becomes

$$S = \frac{1}{2\pi} \int dt dx \left[(\partial_t \theta)^2 - (\partial_x \theta)^2 \right]. \tag{18.3}$$

The current requires some care. The relationship between the real and imaginary time is $it = \tau$. The conservation of imaginary time current reads $\partial_\tau j_\tau^I + \partial_x j_x^I = 0$, while that of the real current reads $\partial_t j_t^R - \partial_x j_x^R = 0$. This implies

$$j_t^R = i j_\tau^I = \frac{1}{\pi} \partial_x \theta, \ j_x^R = j_x^I = \frac{i}{\pi} \partial_\tau \theta = \frac{1}{\pi} \partial_t \theta. \tag{18.4}$$

Once we have an action, a Hamiltonian can be useful. A canonical momentum is $\Pi = \frac{1}{\pi} \partial_t \theta$, and the Hamiltonian is

$$H = \int dx \left[\frac{\pi}{2} \Pi^2 + \frac{1}{2\pi} (\partial_x \theta)^2 \right]. \tag{18.5}$$

The momentum and the "coordinate" satisfy the usual commutation relations

$$[\theta(x) \Pi(y)] = i\delta(x-y). \tag{18.6}$$

It is customary in the applications of the technique to introduce another variable

$$\Pi = \frac{\partial_x \varphi}{\pi}. \tag{18.7}$$

Then we find

$$H = \frac{1}{2\pi} \int dx \left[(\partial_x \theta)^2 + (\partial_x \varphi)^2 \right], \ [\varphi(x)\theta(y)] = -\frac{i\pi}{2} \operatorname{sign}(x-y) \tag{18.8}$$

The usefulness of the variable θ becomes apparent if one notices that θ and ϕ satisfy the following equations

$$\partial_x \theta = \partial_t \varphi, \ \partial_t \theta = \partial_x \varphi. \tag{18.9}$$

The first one follows from the definition of course, and the second one follows from the equations of motion. We now introduce the following left and right densities $\Phi_L = \frac{\theta+\varphi}{2}$, $\Phi_R = \frac{\theta-\varphi}{2}$ which satisfy

$$(\partial_x + \partial_t)\,\Phi_R = 0,\ (\partial_t - \partial_x)\,\Phi_L = 0. \tag{18.10}$$

These guys satisfy the commutation relations

$$[\Phi_R \Phi_R] = \frac{i\pi}{4}\ \mathrm{sign}(x-y),\ [\Phi_L \Phi_L] = -\frac{i\pi}{4}\ \mathrm{sign}(x-y),\ [\Phi_R \Phi_L] = 0. \tag{18.11}$$

The density of electrons can be split into left and right moving parts

$$\rho = j_t^R = \frac{1}{\pi}\partial_x \theta = \frac{1}{\pi}\left(\partial_x \Phi_L + \partial_x \Phi_R\right). \tag{18.12}$$

When expressed in terms of right and left moving modes, the Hamiltonian becomes

$$H = \frac{1}{\pi}\int dx\ \left[(\partial_x \Phi_R)^2 + (\partial_x \Phi_L)^2\right]. \tag{18.13}$$

19 Interactions

The bosonization formula becomes especially useful if we consider interacting electrons. Consider one dimensional spinless electrons interacting via the density-density interaction (such as Coulomb interaction)

$$H = \sum_p \frac{p^2}{2m} a_p^\dagger a_p + \sum_{q,k_1,k_2} U(q)\ a_{k_1+q}^\dagger a_{k_1} a_{k_2-q}^\dagger a_{k_2}. \tag{19.1}$$

In this form the problem of interacting electrons seems completely intractable. However, it is possible to re-write it in its bosonic version, where it will become completely solvable.

It is natural that only those momenta which are around k_F or $-k_F$ contribute to physical interactions. We also assume that $U(q)$ does not change very much over the range of momenta relevant for this problem. In terms of the Dirac fields which capture the physics around the Fermi points

$$a(x) = e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x),\ a^\dagger = e^{-ik_F x}\psi_R^\dagger(x) + e^{ik_F x}\psi_L^\dagger(x) \tag{19.2}$$

The interaction takes the form

$$U(0)\left(\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L\right)^2 + \left(\psi_R^\dagger \psi_L \psi_L^\dagger \psi_R\ U(2k_F) + \psi_L^\dagger \psi_R \psi_R^\dagger \psi_L\ U(-2k_F)\right). \tag{19.3}$$

This can be brought to the form

$$\frac{g_1}{\pi}\left(\psi_R^\dagger\psi_R\psi_R^\dagger\psi_R+\psi_L^\dagger\psi_L\psi_L^\dagger\psi_L\right)+\frac{2g_2}{\pi}\psi_R^\dagger\psi_R\psi_L^\dagger\psi_L. \tag{19.4}$$

The first term is referred to as forward scattering, and the second as dispersion.

Remarkable things start happening if we now use the bosonization formula

$$\bar\psi_L\psi_L=\frac{1}{\pi}\partial_x\Phi_L,\ \bar\psi_R\psi=\frac{1}{\pi}\partial_x\Phi_R. \tag{19.5}$$

Then the total Hamiltonian becomes

$$H=\frac{1}{\pi}\int dx\left[(1+g_1)\left\{(\partial_x\Phi_L)^2+(\partial_x\Phi_R)^2\right\}+2g_2\partial_x\Phi_L\partial_x\Phi_R\right]. \tag{19.6}$$

This Hamiltonian is still quadratic and therefore completely solvable. Consequently, it is possible to solve exactly the problem of interacting electrons in 1 dimensional space. Going back to the variables φ and θ we find

$$H=\frac{1}{2\pi}\int dx\left[(1+g_1+g_2)\left(\partial_x\theta\right)^2+(1+g_1-g_2)\left(\partial_x\varphi\right)^2\right]. \tag{19.7}$$

It is customary to introduce two parameters, the Luttinger parameter g and the speed of sound v, in such a way that

$$H=\frac{v}{2\pi}\int dx\left[\frac{1}{g}\left(\partial_x\theta\right)^2+g\left(\partial_x\varphi\right)^2\right],\ v=\sqrt{(1+g_1)^2-g_2^2},\ g=\sqrt{\frac{1+g_1-g_2}{1+g_1+g_2}}. \tag{19.8}$$

For repulsive interactions $g_2>0$ and $g<1$. It is instructive now to go back to the action. To do that, we use the commutation relations to find the connection between momentum and coordinate $\partial_t\theta=vg\partial_x\varphi$ and therefore

$$S=\frac{1}{\pi}\partial_x\varphi\ \partial_t\theta-H=\frac{1}{2\pi g}\int dxdt\left[\frac{1}{v}\left(\partial_t\theta\right)^2-v\left(\partial_x\theta\right)^2\right]. \tag{19.9}$$

It is clear from here that v is the wave velocity, while g is a nontrivial parameter related to the interaction strength.

Alternatively, we could treat φ as a basic variable. The variable conjugate to φ is $\partial_x\theta/\pi$. We can now use the equation of motion

$$\partial_t\varphi=\frac{v}{g}\partial_x\theta,\ \partial_t\theta=vg\partial_x\varphi \tag{19.10}$$

and write down an alternative ("dual") action, written in terms of the dual variable φ

$$S=\frac{g}{2\pi}\int dxdt\left[\frac{1}{v}\left(\partial_t\varphi\right)^2-v\left(\partial_x\varphi\right)^2\right]. \tag{19.11}$$

This action differs from (19.9) only by redefining $g \to 1/g$. Therefore, there is a deep connection between the theories with $g > 1$ and $g < 1$, which we will explore below.

The actions (19.9) and (19.11) completely capture the physics of interacting 1D electron gas.

It is instructive to rewrite the basic equations of motion (19.10) in the imaginary time (by the appropriate rescaling of x we can get rid of the velocity v)

$$\frac{1}{g}\partial_x\theta = i\partial_\tau\varphi, \ \partial_\tau\theta = -ig\partial_x\varphi. \tag{19.12}$$

These can be conveniently written as

$$\partial_\mu\theta = -ig\epsilon_{\mu\nu}\partial_\nu\varphi. \tag{19.13}$$

It follows from here that $j^\mu = \frac{g}{\pi}\partial_\mu\varphi$. What is more remarkable however is that the axial current now reads

$$\tilde{j}^\mu = \frac{g}{\pi}\epsilon_{\mu\nu}\partial_\nu\varphi \tag{19.14}$$

The basic action together with the external EM field reads (in the imaginary time)

$$S = \int dxd\tau\left[\frac{1}{2\pi g}\left(\partial_\mu\theta\right)^2 - \frac{ie}{\pi}\epsilon_{\mu\nu}A_\mu\partial_\nu\theta\right]. \tag{19.15}$$

From here the axial anomaly reads

$$\partial_\mu\tilde{j}_\mu = -\frac{eg}{\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu. \tag{19.16}$$

The axial anomaly implies

$$\partial_\mu\tilde{j}^\mu = \frac{g}{\pi}\epsilon_{\mu\nu}\partial_\mu\partial_\nu\varphi = -\frac{eg}{\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu. \tag{19.17}$$

This could seem contradictory since $\epsilon_{\mu\nu}\partial_\mu\partial_\nu = 0$. However, it is well known that if φ is not a single valued function, there is no contradiction in (19.17). Indeed, we find

$$\int dxd\tau\partial_\mu\tilde{j}_\mu = \frac{g}{\pi}\int ds^\mu\partial_\mu\varphi = 2ng \tag{19.18}$$

where n is the number of particles transferred from left to right. Therefore, φ is allowed to be discontinuous, up to factors of $2\pi n$. Borrowing terminology from string theorists, we say that φ is a compact variable.

20 The Luttinger liquid

The electron liquid described by (19.9)-(19.11) is called the Luttinger liquid. Consider a Luttinger liquid in an electric field described by (19.15). It is clear that the "anomaly" now reads

$$\partial_\mu \tilde{j}^\mu = -\frac{eg}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}. \tag{20.1}$$

Therefore, if we turn on a field which normally would transfer one unit of electric charge from left to right, this time it will transfer ge charge from left to right. We deduce that the Luttinger liquid has fractionally charged excitations. These were in fact observed experimentally (the edge of a quantum Hall droplet).

Consider a one dimensional wire where a potential U is applied to the left end. It creates an excess of right moving electrons. If the electrons are not interacting, the current can be easily found

$$I = nev = \frac{eU}{2\pi v_F}v_F e = \frac{e^2}{2\pi}U. \tag{20.2}$$

(The particle number is calculated by noting that $eU = v_F\,(p - p_F) = v_F\frac{2\pi N}{L}$). Restoring the $\hbar$ to the formula, it implies that the conductance of a one dimensional wire is $G = \frac{e^2}{2\pi\hbar}$. Now if the particles are not electrons but fractionally charged quasiparticles, the formula changes to

$$G = g\frac{e^2}{2\pi\hbar}. \tag{20.3}$$

To see that, we make use of the Hamiltonian (19.8). Introduce the left and right moving densities

$$n_R = \frac{\partial_x\theta - g\partial_x\varphi}{2\pi} = \frac{\partial_x\phi_R}{\pi},\; n_L = \frac{\partial_x\theta + g\partial_x\varphi}{2\pi} = \frac{\partial_x\phi_L}{\pi}. \tag{20.4}$$

They do not coincide with the densities of the free electron $\partial_x\Phi_{R,L}$. These guys satisfy the commutation relations

$$[\phi_R, \phi_R] = \frac{i\pi g}{4}\mathrm{sign}\,(x-y)\,,\; [\phi_L, \phi_L] = -\frac{i\pi g}{4}\mathrm{sign}\,(x-y)\,. \tag{20.5}$$

In terms of these, the Hamiltonian (19.8) becomes

$$H = \frac{v\pi}{g}\int dx\,\left[n_R^2 + n_L^2\right]. \tag{20.6}$$

Imagine that the chemical potential of the right moving particles was lowered by the amount $A_0 = U$ (recall that A_0 can be thought of as an electrostatic potential), which modifies the Hamiltonian in the following way

$$H = \frac{v\pi}{g}\int dx\,\left[n_R^2 + n_L^2\right] - \int dx\; eUn_R. \tag{20.7}$$

Equilibrium density is the one minimizing the Hamiltonian, and it is given by

$$n_R = \frac{g}{2\pi v} eU. \tag{20.8}$$

Therefore the (physical) current is

$$I = ven_R = g\frac{e^2}{2\pi}U \tag{20.9}$$

as was claimed above.

A better way to understand the Luttinger liquid conductance (due to Matveev and Glazman) is to apply electric field localized in space in an area of the length L

$$E = -U'(x)) \sin \omega t. \tag{20.10}$$

Upon substitution into the Hamiltonian (20.7), we find the equations of motion

$$\partial_t \phi_R + v\partial_x \phi_L = \frac{eUg}{2} \sin(\omega t) \tag{20.11}$$

The current (electron flux) calculated with the help of this equation reads

$$I = ev\frac{iq\phi_R}{\pi} = \frac{e^2 gv}{2\pi} \int \frac{dq}{2\pi} \frac{qU(q)}{\omega_0 + vq}. \tag{20.12}$$

Taking the limit $\omega_0 \to 0$, we find

$$I = \frac{e^2}{2\pi} g \int \frac{dq}{2\pi} U(q) = \frac{e^2}{2\pi} gU(0) \tag{20.13}$$

This assumes, however, that we are probing the flux of particles at distances much larger than v/ω_0, which goes to infinity as $\omega_0 \to 0$.

Finally, we would like to figure out how to identify the left and right moving electrons ψ_L and ψ_R in terms of bosonic fields. We note that $e^{-i\varphi}$ creates an electron. This is because $[-i\varphi(x), \theta(y)] = -\frac{\pi}{2}\text{sign}(x-y)$, so that $e^{-i\varphi(x)}$ increases $\theta(y)$ by the amount $\pi/2$ if $y > x$ and decreases it by the same amount if $y < x$. $\partial_x\theta(y)$ changes as $\partial_x\theta(x) \to \partial_x\theta(x) + \pi\delta(x-y)$. The density of electrons is $\partial_x\theta/\pi$ which completes the proof. However, $e^{-i\varphi}$ is a bosonic operator. To make it fermionic, we add to it a Jordan-Wigner string

$$\psi_R = e^{-i\varphi + i\theta}, \;\; \psi_L = e^{-i\varphi - i\theta}. \tag{20.14}$$

Correspondingly,

$$\psi_R^\dagger = e^{i\varphi - i\theta}, \;\; \psi_L^\dagger = e^{i\varphi + i\theta}. \tag{20.15}$$