

# Advanced Statistical Mechanics

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Week 6

## 14 1D Free Electron Gas

1D (spinless) electrons are described by

$$H = \sum_k \frac{k^2}{2m} a_k^\dagger a_k. \quad (14.1)$$

Suppose all the energy levels are filled with electrons up to the energy  $\mu$ . Then the effective Hamiltonian is  $H - \mu \sum_k a_k^\dagger a_k$ . We call  $k_F = \sqrt{2m\mu}$  the Fermi momentum. There are two Fermi points  $+k_F$  and  $-k_F$ . For  $k$  close to the Fermi points we get

$$H = \sum_k \epsilon_k a_k^\dagger a_k, \quad \epsilon_k \approx \frac{k_F}{m} (k - k_F). \quad (14.2)$$

This formally coincides with the spectrum of Dirac operator at  $m = 0$ .  $v_F = k_F/m$  is called the Fermi velocity, it replaces the speed of light in the Dirac equation.

Suppose an electric field  $E(t)$  is applied to this gas. The electrons accelerate according to  $\dot{k} = eE$ , since  $k$  is the momentum. If then  $E(t)$  turns to zero, the change in the momentum of the electrons is equal to

$$\Delta k = e \int dt E = e(A(t) - A(0)), \quad (14.3)$$

where we introduced the gauge potential such that  $E = \frac{d}{dt}A(t)$ . (This corresponds to  $A$  being the covariant vector, as opposed to the contravariant vector according to Landau-Lifshitz conventions).

Let  $A(0) = 0$ . After  $E$  turns to zero,  $A(t)$  becomes a constant which we call  $A$ . As a result, the chemical potential in the right will shift to

$$\mu_R = \mu + v_F \Delta k = \mu + ev_F A, \quad (14.4)$$

while the one of the left gets lowered by the same amount. The total number of electrons on the right increases by

$$\Delta N_R = \frac{L}{2\pi} \Delta k = \frac{L}{2\pi} eA, \quad (14.5)$$

$L$  is the total length of the 1D system, and it decreases by the same amount on the left.

It is instructive to know what exactly happens quantum mechanically. The Schrödinger equation states

$$i\Psi_t = \frac{1}{2m} \left( -i \frac{\partial}{\partial x} + eA(t) \right)^2 \Psi. \quad (14.6)$$

Look for a solution  $\Psi(x) = e^{ikx+f(t)}$ , where

$$f(t) = -\frac{i}{2m} \int dt (k + eA)^2. \quad (14.7)$$

In other words, the energy of the particle grows from  $k^2/2m$  to  $(k + eA)^2/2m$ . After  $A(t)$  stops changing, we could do a gauge transformation  $\psi = e^{-ieAx}\tilde{\psi}$  to get rid of it. This can only be done if  $eAL = 2\pi n$ . The new momentum is then  $k' = k + eA$ , and  $n$  clearly plays the role of the number of particles transferred from left to right.

## 15 Dirac Equation and the Electric Field

The electrons around the Fermi points are equivalent to the Dirac equation. The Dirac equation can be coupled to the electric field via the gauge principle, just as the Schrödinger equation. We find

$$S = \int dxdt [\bar{\psi} (i\partial_\mu - eA_\mu) \gamma^\mu \psi] \quad (15.1)$$

This equation is still gauge and axial symmetric. In particular, the axial current  $\tilde{j}^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$  is still conserved. However  $\tilde{j}^0$  is the difference between left and right moving particles, and that difference should not be conserved. The nonconservation of  $\tilde{j}^\mu$  is referred to as axial anomaly.

Using the result above we see that

$$\int dxdt \partial_\mu \tilde{j}^\mu = \int dx (j^0(t) - j^0(0)) = -\frac{L}{\pi} eA \quad (15.2)$$

We therefore conjecture that

$$\partial_\mu \tilde{j}^\mu = -\frac{e}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (15.3)$$

This however directly contradicts the Dirac equation.

## 16 Axial Anomaly

In this section we resolve this contradiction by showing that while  $\partial_\mu \tilde{j}^\mu = 0$  classically, the correlation function  $\langle \partial_\mu \tilde{j}^\mu \rangle \neq 0$ .

We work in imaginary time. Let us calculate the correlation function  $\langle \partial_\mu \tilde{j}^\mu \rangle$ . In imaginary time  $\tilde{j}^\mu = i\bar{\psi}\gamma^\mu\gamma^5\psi$ , because the symmetry of the action is now  $\psi' = e^{a\gamma^5}\psi$ , where  $a$  is a real parameter.

The correlation  $\langle \bar{\psi}\psi \rangle$  is the (minus) Green's function of the Dirac equation and as such it's equal to  $-\sum_n \bar{\psi}_n \psi_n / E_n$ . Therefore,

$$i \langle \bar{\psi} \gamma^\mu \gamma^5 \psi \rangle = -i \sum_n \frac{\bar{\psi}_n \gamma^\mu \gamma^5 \psi_n}{E_n}. \quad (16.1)$$

Here  $\psi_n$  are the eigenfunctions of the Dirac equation. They satisfy

$$(i\partial_\mu - A_\mu) \gamma^\mu \psi = E_n \psi, \quad (-i\partial_\mu - A_\mu) \bar{\psi} \gamma^\mu = E_n \bar{\psi}. \quad (16.2)$$

This only works in imaginary time, since otherwise the Dirac equation is not hermitian, and  $\bar{\psi}$  does not coincide with  $\psi^*$ . It follows

$$\partial_\mu \langle \tilde{j}^\mu \rangle = 2 \sum_n \bar{\psi}_n \gamma^5 \psi_n. \quad (16.3)$$

Now  $\sum_n \bar{\psi}_n^i(x) \psi_n^j(y) = \delta(x-y) \delta^{ij}$ , where  $i, j$  are spinor indices, so  $\sum_n \bar{\psi}_n(x) \gamma^5 \psi_n(y) = \delta(x-y) \text{tr} \gamma^5$ . The trace of gamma matrix is zero, but the delta function at coinciding points is infinity, so the answer is indefinite. We need to regularize the sum.

We calculate instead

$$\sum_n \bar{\psi}_n \gamma^5 \psi_n e^{-\lambda E_n^2} \quad (16.4)$$

and take the limit  $\lambda \rightarrow 0$ . We note that  $E_n^2$  are the eigenvalues of the following operator

$$((i\partial_\mu - eA_\mu) \gamma^\mu)^2 = (i\partial_\mu \gamma^\mu - eA_\mu \gamma^\mu)^2 + \frac{e}{2} \gamma^5 \epsilon_{\mu\nu} F_{\mu\nu}. \quad (16.5)$$

Therefore, the sum in (16.4) can be calculated as

$$-\lambda \epsilon_{\mu\nu} F_{\mu\nu} e \int \frac{d^2 p}{4\pi^2} e^{-\lambda p^2} = -\frac{e}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}. \quad (16.6)$$

Finally we find

$$\partial_\mu \langle \tilde{j}^\mu \rangle = -\frac{e}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}, \quad (16.7)$$

which reproduces the ‘‘anomaly’’ (15.3).

It is interesting to note that the massless Dirac equation is chiral. That means that if  $\psi_n$  is an eigenfunction,  $\gamma^5 \psi_n$  is also an eigenfunction, but with the eigenvalue  $-E_n$ . It follows from here that  $\int \bar{\psi}_n \gamma^5 \psi_n = 0$ . The exception to this are the zero modes of the Dirac operator, which satisfy  $\gamma^5 \psi = \pm \psi$  depending on whether they are right or left moving. Therefore, the anomaly is equal to double number of right minus the number of left moving zero modes, and this coincides with the number of right moving particles produced in this motion minus the number of left moving particles removed in it.

## 17 Effective Action

We continue to work in imaginary time. We would like to calculate the effect the external field has on electrons, besides the anomaly effect (15.3). For that we calculate the effective action (free energy in statistical mechanics), defined as

$$F = -\log \int \mathcal{D}\psi e^{-\int dx d\tau [\bar{\psi}(i\partial_\mu - eA_\mu)\gamma^\mu\psi]}. \quad (17.8)$$

Any 2D gauge field  $A$  can be decomposed into pure gauge and an axial factor:

$$A_\mu = \partial_\mu \xi + \epsilon_{\mu\nu} \partial_\nu \phi. \quad (17.9)$$

It follows that

$$\frac{\delta F}{\delta \phi} = e \epsilon_{\mu\nu} \partial_\nu \langle \bar{\psi} \gamma^\mu \psi \rangle. \quad (17.10)$$

Now due to the property of the gamma matrices  $\gamma^\mu \gamma^5 = -i\epsilon_{\mu\nu} \gamma^\nu$  we have

$$\bar{\psi} \gamma^\mu \psi = -i\epsilon_{\mu\nu} \bar{\psi} \gamma^\nu \gamma^5 \psi. \quad (17.11)$$

Consequently,

$$\frac{\delta F}{\delta \phi} = -ie \partial_\mu \langle \bar{\psi} \gamma^\mu \gamma^5 \psi \rangle = -\frac{e^2}{\pi} \partial_\mu^2 \phi, \quad (17.12)$$

where in the last equality we used the axial anomaly (15.3) and expressed  $F_{\mu\nu}$  in terms of  $\phi$ .

This allows to conclude that

$$F = \frac{e^2}{2\pi} \int dx d\tau (\partial_\mu \phi)^2. \quad (17.13)$$

The effective action can be conveniently represented as a functional integral over *bosons*,

$$F = -\log \int \mathcal{D}\theta e^{-\int dx d\tau [\frac{1}{8\pi}(\partial_\mu \theta)^2 + \frac{ie}{2\pi} \epsilon_{\mu\nu} A_\mu \partial_\nu \theta]}. \quad (17.14)$$

This formula lies at the heart of bosonization. The Dirac fermions interacting with the electric field in one spacial dimension are equivalent to free bosons with the action

$$S = \frac{1}{8\pi} \int dx d\tau (\partial_\mu \theta)^2 \quad (17.15)$$

This is the canonical form of the bosonized action. The Dirac current, when expressed in terms of  $\theta$ , reads

$$j^\mu = -\frac{i}{2\pi} \epsilon_{\mu\nu} \partial_\nu \theta. \quad (17.16)$$