Advanced Statistical Mechanics

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Week 5

10 2D Dirac Equation

Dirac invented his famous equation trying to write down something first order in derivatives and relativistic.

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0. \tag{10.1}$$

γ^μ are matrices satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g_{\mu\nu}, \tag{10.2}$$

where $g_{\mu\nu} = 0$ if $\mu \neq \nu$ and $g_{00} = 1$, $g_{ii} = -1$, $i > 0$. ψ is a multicomponent spinor.

In four dimensional space γ_μ are 4 by 4 matrices. In the two dimensional space time they are 2 by 2 matrices instead, which can be chosen as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau^1,\ \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\tau^2, \tag{10.3}$$

where τ are the Pauli matrices.

Writing Dirac fermion explicitly as $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, we find

$$\begin{pmatrix} -m & i\partial_0 + i\partial_1 \\ i\partial_0 - i\partial_1 & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0. \tag{10.4}$$

If $m = 0$, then ψ_1 is clearly a function of $x + t$ only, while ψ_2 is a function of $x - t$. That's why they are called left and right Dirac fermions respectively.

Application of a second Dirac operator with the opposite sign of m gives

$$(i\gamma^\mu \partial_\mu + m)\,(i\gamma^\mu \partial_\mu - m) = -\partial_0^2 + \partial_1^2 - m^2. \tag{10.5}$$

Therefore, a solution to the Dirac equation proportional to $e^{ikx - i\omega t}$ will always satisfy

$$\omega^2 = k^2 + m^2. \tag{10.6}$$

The Dirac action is

$$S = \int dx dt\ \left[\bar\psi\,(i\gamma^\mu \partial_\mu - m)\,\psi\right] \tag{10.7}$$

Explicitly, it is

$$S = \int dx dt\ \left[\bar\psi_1\,(i\partial_0 + i\partial_1)\,\psi_2 + \bar\psi_2\,(i\partial_0 - i\partial_1)\,\psi_1 - m\left(\bar\psi_1 \psi_1 + \bar\psi_2 \psi_2\right)\right]. \tag{10.8}$$

The Dirac conjugate spinor satisfies, as a consequence of (10.7),

$$\left(i\gamma^{T\mu} \partial_\mu + m\right)\bar\psi = 0, \tag{10.9}$$

where γ^T are the transposed matrices. It is possible to express $\bar{\psi}$ in terms of ψ^* if one notices

$$(i\gamma^\mu \partial_\mu + m)\,\psi^* = 0. \tag{10.10}$$

Now

$$\gamma^0 \left(i\gamma^{T\mu}\partial_\mu + m\right)\bar{\psi} = (i\gamma^\mu \partial_\mu + m)\,\gamma_0\bar{\psi}, \tag{10.11}$$

therefore

$$\bar{\psi} = \gamma^0\psi^* \tag{10.12}$$

In addition to this, we define a charge conjugate spinor as

$$\psi^C = \gamma^5\psi^*,\ \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tau^3. \tag{10.13}$$

Its main property is that, even though it is proportional to ψ^*, it satisfies

$$(i\gamma^\mu \partial_\mu - m)\,\psi^C = 0, \tag{10.14}$$

same as the original Dirac equation. As a result, we could construct

$$\psi_1^M = \psi + \psi^C,\ \psi_2^M = i\left(\psi - \psi_C\right), \tag{10.15}$$

which are called the Majorana fermions. It is clear that

$$\left(\psi^M\right)^C = \psi^M. \tag{10.16}$$

The Majorana fermions are the closest one can get to real fermions. Indeed, linear combinations $\psi \pm \psi^*$, which are real, would not work as they do not satisfy the Dirac equations. We say that 1 Dirac fermion consists of two Majorana fermions.

The action for a Majorana fermion can be written if one notices that

$$\bar{\psi}^M = \psi^*\gamma^0 = \left(\psi^M\right)^C\gamma^5\gamma^0 = \psi^M\gamma^5\gamma^0. \tag{10.17}$$

We thus get

$$S = \int dxdt\ \psi^M\gamma^5\gamma^0\,(i\gamma^\mu\partial_\mu - m)\,\psi^M \tag{10.18}$$

Writing Majorana fermion explicitly as $\psi^M = \begin{pmatrix} \psi_1^M \\ \psi_2^M \end{pmatrix}$, we find

$$S = \int dxdt\ \left[\psi_1^M\left(i\partial_0 - i\partial_1\right)\psi_1^M + \psi_2^M\left(-i\partial_0 - i\partial_1\right)\psi_2^M - m\left(\psi_1^M\psi_2^M - \psi_2^M\psi_1^M\right)\right]. \tag{10.19}$$

Quite remarkably, this object vanishes unless ψ anticommutes with itself. We conclude that ψ has to be Grassmanian (anticommuting) variable.

11 Dirac Equation in Imaginary Time

If the time is imaginary, Dirac equation becomes

$$\left[-\gamma^0\partial_0 + i\gamma^1\partial_1 - m\right]\psi = 0. \tag{11.1}$$

It is convenient to introduce "imaginary" gamma matrices, which satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}. \tag{11.2}$$

They can be chosen as $\gamma^0 = \tau_x$, $\gamma^1 = \tau_y$, the Pauli matrices. The Dirac equation is then

$$[i\gamma^\mu\partial_\mu + im]\psi = 0 \tag{11.3}$$

It is easy to see that $\bar{\psi} = \psi^*\gamma^5$. The Dirac action is now explicitly

$$S = \int dxdt\ \left[\bar{\psi}_1(i\partial_0 + \partial_1)\psi_2 + \bar{\psi}_2(i\partial_0 - \partial_1)\psi_1 + im\left(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2\right)\right]. \tag{11.4}$$

The charge conjugate condition also takes a slightly different form

$$\psi^C = \gamma^0\psi^*, \tag{11.5}$$

since $\gamma^5\psi^*$ will satisfy the same Dirac equation as ψ. A Majorana fermion is the one whose charge conjugate coincides with the fermion itself. As a result, $\bar{\psi}^M = \psi^*\gamma^5 = \psi^C\gamma^0\gamma^5 = \psi^M\gamma^0\gamma^5$, and

$$S = \int dxdt\ \left[\psi_1^M(-\partial_0 - i\partial_1)\psi_1^M + \psi_2^M(\partial_0 - i\partial_1)\psi_2^M + im\left(\psi_2^M\psi_1^M - \psi_1^M\psi_2^M\right)\right]. \tag{11.6}$$

12 Quantization of the Dirac Equation

12.1 Harmonic Oscillator

Any free field theory is a bunch of noninteracting oscillators with different frequencies. One oscillator can be quantized simply:

$$S = \frac{1}{2}\int dt\ \left[\dot{x}^2 - \omega_0^2 x^2\right]. \tag{12.1}$$

Classically

$$x(t) = \frac{1}{\sqrt{2\omega_0}}\left[ae^{-i\omega_0 t} + a^* e^{i\omega_0 t}\right]. \tag{12.2}$$

Upon quantization a, a^* become creation and annihilation operators a and $a^\dagger$, as obvious by their commutation relations:

$$p = \dot{x} = -i\sqrt{\frac{\omega_0}{2}}\left[a e^{i\omega_0 t} - a^* e^{-i\omega_0 t}\right], \tag{12.3}$$

$$[x, p] = i \ \rightarrow [a, a^\dagger] = 1. \tag{12.4}$$

12.2 Free Boson

The technique to quantize a field theory is to split it into many harmonic oscillators, each of which can be quantized using standard quantum mechanics techniques. No one ever writes the wave function of a field since it is a function of a large number of coordinates and as such is useless. Usually one needs just the spectrum and matrix elements, and for this the creation and annihilation operators suffice.

Let us do it on the example of the simplest field theory

$$S = \frac{1}{2}\int dxdt \ \left[\dot{\phi}^2 - \phi_x^2\right]. \tag{12.5}$$

We expand in Fourier series

$$\phi = \int \frac{dk}{2\pi} \ a_k e^{ikx} \tag{12.6}$$

$$S = \frac{1}{2}\int \frac{dtdk}{2\pi} \ \left[\dot{a}_k \dot{a}_{-k} - k^2 a_k a_{-k}\right]. \tag{12.7}$$

Each of the Fourier modes can be treated independently. Taking into account that $a_k^* = a_{-k}$ and denoting $\psi = a_k$, we write for a mode with a given k,

$$S = \int dt \ \left[\dot{\psi}^* \dot{\psi} - k^2 \psi^* \psi\right], \tag{12.8}$$

and this is the action of two oscillators with the frequency $\omega_0 = |k|$, which can be seen by substituting $\psi = x + iy$.

The easiest way to write down creation and annihilation operators is by writing the solution to the classical equations of motion. Denoting $\omega_k = |k|$ we write

$$\phi = \sum_k \ \frac{1}{\sqrt{2\omega_k}}\left[a_k \ e^{ikx - i\omega_k t} + a_k^\dagger e^{-ikx + i\omega_k t}\right]. \tag{12.9}$$

We say that $a_k^\dagger$ creates bosons with momentum k and frequency ω_k.

The canonical momentum of the field is $\Pi = \dot{\phi}$, and the Hamiltonian can be written as

$$H = \frac{1}{2}\int dx \left[\Pi^2 + \phi_x^2\right] = \frac{1}{2}\int \frac{dk}{2\pi} \ \omega_k \left(a_k^\dagger a_k + a_k a_k^\dagger\right). \tag{12.10}$$

These are a bunch of oscillators with frequencies ω_k.

12.3 Complex Free Boson

A complex free boson has the same action (12.5), except the field is now complex. We call it ψ,

$$S = \int dxdt \, \left[\dot{\psi}^* \dot{\psi} - \psi_x^* \psi_x\right]. \tag{12.11}$$

Clearly it has twice as many degrees of freedom, and it is quantized as in

$$\psi = \sum_k \frac{1}{\sqrt{2\omega_k}} \left[a_k \, e^{ikx - i\omega_k t} + b_k^\dagger e^{-ikx + i\omega_k t}\right]. \tag{12.12}$$

We have two species of particles, which are created by $a^\dagger$ and $b^\dagger$. They are antiparticles to each other. The Hamiltonian is

$$H = \int \frac{dk}{2\pi} \, \omega_k \left[a_p^\dagger a_p + b_p b_p^\dagger\right]. \tag{12.13}$$

12.4 Dirac Equation

Similarly, we look for the classical solution to the Dirac equation

$$\psi = u_k a_k e^{ikx - i\omega_k t} + v_k b_k^\dagger e^{-ikx + i\omega_k t}. \tag{12.14}$$

Here u_k and v_k are 2-dimensional vectors (spinors) which satisfy

$$\left[\gamma^0 \omega_k - \gamma^1 k - m\right] u_k = 0, \; \left[-\gamma^0 \omega_k + \gamma^1 k - m\right] v_k = 0. \tag{12.15}$$

Consequently they satisfy

$$v_k = \gamma^5 u_k^*. \tag{12.16}$$

The condition for these equations to be solvable is, of course,

$$\omega = \omega_k, \; \omega_k = \sqrt{k^2 + m^2}. \tag{12.17}$$

a and $a^\dagger$ are fermion creation and annihilation operators, and so are b, $b^\dagger$.

The Hamiltonian of the Dirac equation can be found in the usual way $\Pi = i\psi^*$, and

$$H = \int dx \, \left[-\Pi \gamma^2 \partial_1 \psi - im \Pi \gamma^0 \psi.\right] = \sum_k \omega_k \left[a_k^\dagger a_k - b_k b_k^\dagger\right]. \tag{12.18}$$

For this to be positive, a and b have to anticommute.

$a_k^\dagger$ creates particles, $b^\dagger$ creates holes. There are two ways to create a state with energy ω_k: one is to create a particle, the other a hole.

Majorana fermions satisfy $\psi^C = \psi$, consequently (taking into account (12.16))

$$\psi = u_k a_k e^{ikx - i\omega_k t} + v_k a_k^\dagger e^{-ikx + i\omega_k t}. \tag{12.19}$$

The Hamiltonian becomes

$$H = \sum_k \omega_k \left[a_k^\dagger a_k - a_k a_k^\dagger \right]. \tag{12.20}$$

We recognize the Hamiltonian of the Ising model. The Ising model is equivalent to a Majorana fermion.

13 Massless Dirac Equation

The massless Dirac fermion has the action

$$S = \int dxdt \int dxdt \ \left[\psi_2^* \left(i\partial_0 + i\partial_1 \right) \psi_2 + \psi_1^* \left(i\partial_0 - i\partial_1 \right) \psi_1 \right]. \tag{13.1}$$

Upon solving the classical equation of motion, or upon quantization, we discover that ψ_1 depends only on the combination $x + t$, whereas ψ_2 depends on $x - t$. We call them left moving and right moving spinors respectively.

$$\psi_1 = \sum_{k>0} a_k e^{-ik(x+t)} + b_k^\dagger e^{ik(x+t)}, \psi_2 = \sum_{k>0} a_{-k} e^{ik(x-t)} + b_{-k}^\dagger e^{-ik(x-t)}. \tag{13.2}$$

$a_k^\dagger$ creates left moving particles, $a_{-k}^\dagger$ creates right moving particles, and $b_k^\dagger$ create left or right moving antiparticles.

Dirac action (even with a mass) is invariant under $\psi' = e^{ia}\psi$. This leads to conserved current

$$j^\mu = \bar{\psi} \gamma^\mu \psi. \tag{13.3}$$

It is straightforward to write down the current in terms of Dirac spinors:

$$j^0 = \psi_1^* \psi_1 + \psi_2^* \psi_2, \ j^1 = - \left(\psi_1^* \psi_1 - \psi_2^* \psi_2 \right). \tag{13.4}$$

The first term is the particle number, the second is their flux.

Additionally, the massless Dirac action has another symmetry, called the axial symmetry, $\psi' = e^{ia\gamma^5}\psi$. It leads to conserved current, called axial current

$$\tilde{j}^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi. \tag{13.5}$$

In components,

$$\tilde{j}^0 = \psi_1^* \psi_1 - \psi_2^* \psi_2, \ \tilde{j}^1 = - \left(\psi_1^* \psi_1 + \psi_2^* \psi_2 \right) \tag{13.6}$$

It is customary to define the left and right currents as $J_1 = \psi_1^* \psi_1$ and $J_2 = \psi_2^* \psi_2$. As a consequence of the conservation of the normal and axial currents,

$$(\partial_0 - \partial_1)\, J_1 = 0,\ (\partial_0 + \partial_1)\, J_2 = 0. \tag{13.7}$$

At this point it is convenient to define light-cone coordinates $s = x + t$ and $\bar{s} = x - t$. Then

$$\bar{\partial} J_1 = 0,\ \partial J_2 = 0. \tag{13.8}$$