# Advanced Statistical Mechanics

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Week 3

# 6 2D Ising Model: Peierls Argument

$2D$ Ising model is defined on a 2D lattice.

$$Z = \sum_{\sigma_i=\pm 1} e^{\beta H},\ H = \sum_{i,\mu} \sigma_i \sigma_{i+\mu}, \tag{6.1}$$

where $i$ is the position of the spin and $i + \mu$ denotes either vertical or horizontal neighbor of $i$.

Peierls argument in this case says that at large $\beta$ the spins will order. Indeed, a droplet of negative spins in the sea of positive spins has relative energy $2\beta L$ and relative contribution to $Z$ as $e^{-2\beta L}$, where $L$ is the length of the droplet's boundary. On the other hand, there are $c^L$ ways to position a droplet. This produces a contribution to $Z$ of the order of $e^{L\log\ c - 2\beta L}$ which, at large $\beta$, will always be small. In other words, the free energy of the droplet is

$$F = E - TS = 2\beta L - L \log c \tag{6.2}$$

and its minimum at large beta is achieved at small $L$.

At critical $\beta$ there's a phase transition between ordered and disordered spins.

# 7 Transfer Matrix and the Hamiltonian

We designate one direction as space, the other as time. We will also allow for the bond strengths $\beta$ to be different along these two directions. We write the partition function as

$$Z = T^N \tag{7.3}$$

where $T$ is a $2^N$ by $2^N$ matrix. Its explicit form is

$$T_{\tilde\sigma,\sigma} = a^N \sum_{\{\sigma'\}=\pm 1} \prod_i \left[e^{\gamma \tau_i^1}\right]_{\tilde\sigma_i \sigma'_i} \left[\prod_i e^{\beta \tau_i^3 \tau_{i+1}^3}\right]_{\{\sigma'\}\{\sigma\}}, \tag{7.4}$$

where $\tau_i^1$,$\tau_i^3$ are the appropriate Pauli matrix which act on the $i$-th Ising spin, and $\sinh\gamma = \exp(-\beta)/\sqrt{2\sinh(2\beta)}$, just as in 1D.

We would like to write this down as $T = \exp(-NH)$. In general it's not easy to find $H$, but at least in one regime this is very easy. Take $\gamma \ll 1$, $\beta \ll 1$. Since $\gamma \propto \exp(-2\beta)$, this is only possible if the bond strengths are different along space and time direction. Then

$$H = -\gamma \sum_i \tau_i^1 - \beta \sum_i \tau_i^3 \tau_{i+1}^3. \tag{7.5}$$

This Hamiltonian is called the 1D transfer field Ising model. The reason for it is simple: this is a 1D chain of quantum spins in the magnetic field perpendicular to them.

# 8 Ordered and Disordered Regimes

If $\gamma \ll \beta$, then as a first approximation the ground state is

$$|0\rangle = \prod_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \text{or } |0\rangle = \prod_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i. \tag{8.6}$$

The first excited state is the one where spins to the left of a certain point point up and the one to the right point down.

$$|y\rangle = \prod_{i<y} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \prod_{i\geq y} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i. \tag{8.7}$$

Now let us add the $\gamma$ term as a small perturbation. Its matrix elements are

$$\langle y| \gamma \sum_i \tau_i^1 |y'\rangle = -\gamma\delta_{y,y'+1} - \gamma\delta_{y,y'-1} \tag{8.8}$$

Diagonalizing this gives the spectrum

$$E(k) - E(0) = 2\beta - 2\gamma\cos(k) = 2\beta - 2\gamma + 4\gamma\sin^2\left(\frac{k}{2}\right). \tag{8.9}$$

This gives a gap of $2\beta - 2\gamma$, so the spins are ordered (assuming it survives at higher order, which it does as the exact solution will show).

Analogous calculation at $\gamma \gg \beta$ gives a gap as well, $2\gamma - 2\beta$, but the spins are disordered.

# 9 Jordan-Wigner transformation

Now we will solve the Hamiltonian (7.5) exactly. First of all we do a unitary transformation so that

$$H = -\gamma\sum_i \tau_i^3 - \beta\sum_i \tau_i^1\tau_{i+1}^1. \tag{9.10}$$

Corresponding transformation is given by $U = \frac{1}{\sqrt{2}}(1 + i\tau^2)$, and

$$U^\dagger\tau^3U = -\tau^1,\ U^\dagger\tau^1U = \tau^3. \tag{9.11}$$

It is convenient to introduce

$$b = \frac{1}{2}\left(\tau^1 + i\tau^2\right),\ b^\dagger = \frac{1}{2}\left(\tau^1 - i\tau^2\right) \tag{9.12}$$

They satisfy

$$b^2 = b^{\dagger 2} = 0, \;\; b^\dagger b + b b^\dagger = 1 \tag{9.13}$$

In this way they look like creation and annihilation operators.

Now we introduce the Jordan-Wigner string

$$a_j = e^{i\pi \sum_{k<j} b_k^\dagger b_k} b_j, \;\; a_j^\dagger = e^{-i\pi \sum_{k<j} b_k^\dagger b_k} b_j^\dagger. \tag{9.14}$$

Then not only $a_i$ obey creation and annihilation relations on a given site, but also

$$a_i^\dagger a_j + a_j a_i^\dagger = 0, \;\; i \neq j, \tag{9.15}$$

and so do $a_i$, $a_j$ and $a_i^\dagger$, $a_j^\dagger$. Therefore, these are now fermions, called Jordan-Wigner fermions. Therefore, the Hamiltonian becomes (taking into account the Jordan-Wigner string)

$$H = \gamma \sum_i \left( a_i^\dagger a_i - a_i a_i^\dagger \right) - \beta \sum_i \left( a_i^\dagger - a_i \right) \left( a_{i+1}^\dagger + a_{i+1} \right) \tag{9.16}$$

This is the so-called Bogoliubov Hamiltonian, studied in superconductivity. To diagonalize it, we employ the Bogoliubov transformations,

$$\begin{pmatrix} c \\ c^\dagger \end{pmatrix} = U \begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{9.17}$$

Here $U$ has to be unitary, $U^\dagger U = 1$ as a consequence of the anticommutation relations.

In this basis,

$$H = \begin{pmatrix} \gamma + \frac{\beta}{2}\Delta & \frac{\beta}{2}\Gamma \\ -\frac{\beta}{2}\Gamma & -\gamma - \frac{\beta}{2}\Delta \end{pmatrix} = \left( \gamma + \frac{\beta}{2}\Delta \right) \Sigma_3 + i\frac{\beta}{2}\Gamma \Sigma_2, \tag{9.18}$$

where $\Delta$ and $\Gamma$ are matrices such that

$$\Delta_{ij} = \delta_{i,j+1} + \delta_{i,j-1}, \;\; \Gamma_{ij} = \delta_{i,j-1} - \delta_{i,j+1}, \tag{9.19}$$

and $\Sigma_3$, $\Sigma_2$ are Pauli matrices. Our task is to diagonalize $H$, after which it can be written as

$$H = \sum_i \left[ h_i c_i^\dagger c_i - h_i c_i c_i^\dagger \right], \tag{9.20}$$

where $h_i$ are eigenvalues of $H$ (more precisely, the absolute value of the eigenvalues).

To find $h_i$, we will use a series of unitary transformations.

We first rotate $\Sigma_3$ into $\Sigma_1$ by employing $U = \frac{1}{\sqrt{2}}(1 + i\Sigma_2)$ and $H' = U^\dagger H U$. We find

$$H = \left( \gamma + \frac{\beta}{2}\Delta \right) \Sigma_1 + i\frac{\beta}{2}\Gamma \Sigma_2. \tag{9.21}$$

The eigenvalue equations become

$$hs_i = \gamma t_i + \beta t_{i+1}, \tag{9.22}$$

$$ht_i = \gamma s_i + \beta t_{i-1}. \tag{9.23}$$

These can be solved in terms of plane waves, to get

$$h(k) = \pm\sqrt{(\gamma-\beta)^2 + 4\gamma\beta\sin^2\left(\frac{k}{2}\right)}. \tag{9.24}$$

We fill all the negative $h(k)$ states with fermions to lower the total energy as much as we can. Then the excitations are either filling the states with negative $h(k)$ or vacating the states with positive $k$. The excitation spectrum is then $2h(k)$. It is easy to check that in the limit $\gamma \ll \beta$ it crosses over to (8.9).

Interesting to note that at small $k$ it has a relativistic form,

$$h^2(k) = (\gamma-\beta)^2 + \gamma\beta k^2. \tag{9.25}$$

This is the spectrum of particles of mass $m = |\gamma - \beta|$.

Finally, the correlation length is the gap and is given by

$$\xi \propto \frac{1}{|\gamma-\beta|} \propto \frac{1}{m}. \tag{9.26}$$

It diverges at $\gamma = \beta$. It is customary, if the correlation function is divergent at some critical $T_c$, to denote the power of its divergence as $\nu$:

$$\xi \propto \frac{1}{(T-T_c)^\nu}. \tag{9.27}$$

We see that for the 2D Ising model $\nu = 1$. This is one of Onsager'z exact results. Another result for the free energy easily follows as well. The free energy is given by

$$F = -T\log Z. \tag{9.28}$$

Since $Z$ is proportional to $\exp(-NE_0)$ where $E_0$ is the ground state energy and $N$ is the length of the Ising model in the "time" direction, that gives

$$F = TNE_0 \tag{9.29}$$

In other words, the ground state energy in quantum mechanics is the same as the free energy in statistical mechanics. In this problem

$$E_0 \propto -\int dk \sqrt{(\gamma-\beta)^2 + 4\gamma\beta \sin^2\left(\frac{k}{2}\right)} \tag{9.30}$$

An interesting quantity is the specific heat.

$$c = -T\frac{\partial^2 F}{dT^2} \tag{9.31}$$

To find it we need to differentiate $E_0$ with respect to $T$, remembering that $\gamma$ and $\beta$ are both temperature dependent. Differentiating $E_0$ with respect to them we find

$$c \propto \log\left(\frac{1}{|\gamma-\beta|}\right). \tag{9.32}$$

This is another of Onsager's famous results. A standard notation in statistical mechanics is

$$c \propto \frac{1}{|T-T_c|^\alpha} \tag{9.33}$$

From here it follows that $\alpha = 0$.