Advanced Statistical Mechanics

Victor Gurarie

1 Particles and Fields

First we review the dynamics of a particle. A particle whose coordinate is $x(t)$ moves along a trajectory which is determined by the minimization of its action $\delta S/\delta x(t) = 0$. An action can be expressed in terms of the Lagrange function as in

$$S = \int_{t_1}^{t_2} dt \; L(x, \dot{x}). \tag{1.1}$$

Minimizing it with respect to $x(t)$ we find the Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}. \tag{1.2}$$

A typical Lagrange function for a point particle is

$$L = \frac{1}{2}\dot{x}^2 - U(x), \tag{1.3}$$

and the equations of motion follow

$$\ddot{x} = -\frac{dU}{dx}. \tag{1.4}$$

A particle living in a multidimensional space is described by its coordinates $x_\mu(t)$, $\mu = 1, 2, \ldots$, and $L = L(x_\mu, \dot{x}_\mu)$. The Lagrange equations follow from minimizing the action are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\mu} = \frac{\partial L}{\partial x_\mu}, \;\; \mu = 1, 2, \ldots. \tag{1.5}$$

Several particles would be described by several coordinates $\vec{\mathbf{x}}_a$, where a goes over different particles. From this point of view, having many interacting particles is the same as having just one particle moving in a multidimensional space.

A field (such as electromagnetic field) is a function of positions in space and time, $\phi(x_2, x_3, \ldots, t)$. It also has an action $S = \int dt \; L$. The Lagrange function L in turn for a local field theory can be expressed as an integral over space of the Lagrangian $\mathcal{L}$:

$$L = \int d^d x \; \mathcal{L}. \tag{1.6}$$

We denote $x_1 \equiv t$. Then $\mathcal{L}(\phi, \partial_\mu \phi)$ is the density of the Lagrange function or a Lagrangian. Lagrange equations can be derived from $\delta S/\delta\phi(t, x) = 0$, and give

$$\sum_\mu \frac{d}{dx_\mu}\frac{\partial L}{\partial\left(\partial_\mu \phi\right)} = \frac{\partial \mathcal{L}}{\partial \phi}. \tag{1.7}$$

This is a typical field lagrangian, a generalization of the Lagrangian for a particle, $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\sum_i(\partial_i\phi)^2 - U(\phi)$. $i = 2, 3, \ldots$ goes over spacial coordinates. Its equation of motion follows

$$\frac{\partial^2\phi}{\partial t^2} - \sum_i \frac{\partial^2\phi}{\partial x_i^2} = -\frac{dU}{d\phi} \tag{1.8}$$

A particle can be thought of as a field living in a one dimensional space which consists of t only: $\phi(x_1)$. That's why mechanics of a particle is often referred to as one dimensional field theory.

A field can on the other hand be thought of as an enormous bunch of particles: for each value of coordinates x_i $\phi(t)_{x_i}$ defines a particle. Or it can be thought of as a particle in an infinite dimensional space. That's why sometimes we say that a field theory is a mechanics with an infinite dimensional degrees of freedom.

Hamiltonian formalism can also be useful. The momentum is defined as $p = \frac{dL}{d\dot{x}}$ and the Hamilton function is defined as

$$H(x, p) = \dot{x}p - L, \tag{1.9}$$

where $\dot{x}$ is expressed in terms of p. For a typical Lagrange function (1.3) $H = \frac{1}{2}p^2 + U(x)$. For many particles (or in multidimensional space) $p_\mu = \frac{dL}{d\dot{x}_\mu}$ and $H = \sum_\mu \dot{x}_\mu p_\mu - L$. Hamilton equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p}, \ \dot{p} = -\frac{\partial H}{\partial x}. \tag{1.10}$$

There exists a notion of Poisson brackets.

$$\{A(x, p), B(x, p)\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} \tag{1.11}$$

For any quantity $A(x, p)$,

$$\dot{A}(x, p) = \{H, A\} \tag{1.12}$$

A generalization of that to the case of fields is straightforward. The momentum density is $\Pi = \delta\mathcal{L}/\delta\dot{\phi}$. The Hamilton function density is

$$H = \Pi\dot{\phi} - \mathcal{L} \tag{1.13}$$

and the Poisson bracket is defined as

$$\{A(\phi, \Pi), B(\phi, \Pi)\} = \int d^d x \left[\frac{\delta A}{\delta\Pi(x)}\frac{\delta B}{\delta\phi(x)} - \frac{\delta A}{\delta\phi(x)}\frac{\delta B}{\delta\Phi(x)}\right] \tag{1.14}$$

2 Noether Theorem

Noether theorem consists of two parts. Part I claims that for every symmetry there is a conservation law. Part II claims that the conserved quantity generates the symmetry transformations. Let's review these.

Consider a free particle Lagrangian $L = \frac{1}{2}\dot{x}^2$. It's invariant under "field shifts" in the field theory language (or translations, in the usual particle language) $x \to x + a$. To derive the conserved quantity we take an infinitesimal version of the transformation $\tilde{x} = x + \epsilon$ and make ϵ an arbitrary function of time t. We find

$$\delta S = S[\tilde{x}] - S[x] \approx \int_{t_1}^{t_2} dt\ \dot{x}\dot{\epsilon} = \dot{x}\epsilon|_{t=t_1}^{t=t_2} - \int dt\ \epsilon \frac{dx^2}{dt^2}. \tag{2.1}$$

The second term vanishes as long as x satisfies equations of motion. The first term, if ϵ is chosen to be a constant (thus representing a constant shift of x) vanishes only if $P = \dot{x}$ does not change with time. P in this case is of course the momentum of the particle, but in general we will call it "conserved charge".

The transformation law we considered above is the "field transformation". Now we consider a "coordinate transformation", $t \to t + a$. It is of course a symmetry not only for a free particle but also for a general lagrangian as long as it does not depend on time explicitly (for example, because of the time dependent potential a particle might feel). Take an time dependent infinitesimal version of this transformation $\tau = t + \delta t(t)$. Take the time shifted trajectory $\tilde{x}(\tau) = x(t)$. It is clear that as a consequence of the action not depending on time

$$\delta S = \int_{t_1+\delta t}^{t_2+\delta t} d\tau\ L\left(\tilde{x}(\tau), \frac{d\tilde{x}(\tau)}{d\tau}\right) - \int_{t_1}^{t_2} dt\ L(x(t), \dot{x}(t)) = 0 \tag{2.2}$$

as long as $\delta(t) = \text{const}$. On the other hand, substituting $\tilde{x}(\tau) = x(t)$ and $\frac{d\tilde{x}}{d\tau} = \frac{dx(t)}{dt}\left(1 - \frac{d}{dt}\delta t\right)$, we find

$$\int_{t_1}^{t_2} dt\ \frac{d}{dt}(\delta t)\left[L - \frac{\partial L}{\partial \dot{x}}\dot{x}\right] = \delta t\ \left[L - \frac{\partial L}{\partial \dot{x}}\dot{x}\right]\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\ \delta t\left[\frac{\partial L}{\partial x}\dot{x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\dot{x}\right]. \tag{2.3}$$

Now we recognize the equation of motion in the second term on the right hand side, and from here we deduce that if $\delta t = \text{const}$, then the energy $E = \frac{\partial L}{\partial \dot{x}}\dot{x} - L = \frac{1}{2}\dot{x}^2 + U(x)$ is conserved.

The second part of Noether theorem states that the Poisson bracket of the conserved quantity with a field generates the transformation of that field. What it formally means is, if $Q(p,x)$ is a quantity conserved as a consequence of a certain transformation, then this transformation is given by $\tilde{A} = A + \epsilon\{Q, A\}$, where ϵ is a infinitesimal number. Indeed,

$$\{P, x\} = 1,\ \{E, x\} = p = \dot{x} \tag{2.4}$$

In field theory Noether theorem predicts that a symmetry leads to conserved current, rather than just to a conserved quantity. Take, for example, the Lagrangian $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\sum_i(\partial_i\phi)^2$. It is invariant with respect to the shift of ϕ by an arbitrary number. In a fashion analogous to the case of particles (when the conservation of momentum was derived), we find

$$\delta S = \int d^d x dt\ \left[\dot{\phi}\ \dot{\delta\phi} - \partial_i\phi\ \partial_i\delta\phi\right] = \int ds_\mu\ j_\mu \delta\phi. \tag{2.5}$$

Here $j_0 = \dot{\phi}$, $j_i = -\partial_i\phi$ is the conserved current. This should vanish if $\delta\phi =$ const. By Gauss theorem, the current is conserved $\partial_\mu j_\mu = 0$.

The conserved charge is given by $Q = \int d^d x\ j_0$.

3 Stress-Energy tensor

Consider a field theory i with the action S. Under $y_\mu = x_\mu + \epsilon_\mu(x)$ the action changes as

$$\delta S = \int d^d x\ \partial_\mu \epsilon_\nu T_{\mu\nu} = \int ds_\mu \epsilon_\nu T_{\mu\nu} - \int d^d x \epsilon_\nu \partial_\mu T_{\mu\nu} \tag{3.1}$$

The last term is always zero by equations of motion. Therefore, $j_\mu = \epsilon_\nu T_{\mu\nu}$ is conserved. $T_{\mu\nu}$ is called stress energy tensor.