Problem Set 7

Phys 7240
Due: Apr 13

1 Relevant and Irrelevant Coupling Constants

Consider the following action in a d-dimensional space

$$S = \int d^d x \left[\frac{1}{2} \sum_{\mu=1}^{d} \left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \lambda_1 \phi^4 + \lambda_2 \left(\sum_{\mu=1}^{d} \partial_\mu \partial_\mu \phi \right)^2 \right]. \tag{1.1}$$

Determine which of the coupling constants m, λ_1, and λ_2 are relevant, which ones are irrelevant and which ones are marginal.

2 Continuum Limit

Consider a field theory defined on a d-dimensional lattice.

$$S = \sum_i \sum_{\mu=1,2,3,\ldots,d} \left(\phi(i+e_\mu) - \phi(i)\right)^2. \tag{2.1}$$

Here i labels vertices of the lattice, and $i + e_\mu$ specifies the nearest neighbor of the i-th vertex in the μ-th direction.

1. Substitute a Taylor series expansion

$$\phi(i+e_\mu) = \phi(i) + e_\mu \frac{\partial \phi(i)}{\partial e_\mu} + \ldots \tag{2.2}$$

into Eq. (2.1).

2. Show that Eq. (2.1) can be written as

$$S = a^\gamma \int d^d x \, \sum_{\mu=1}^{d} \left(\partial_\mu \phi\right)^2 + \text{irrelevant terms}. \tag{2.3}$$

where a is the lattice spacing (determine γ).

This is the reason why in both particle physics and condensed matter physics Eq. (2.3) is usually used over the more complicated, but equivalent in the RG sense, Eq. (2.1).

3 Sine-Gordon model

This theory set in the 2-dimensional space is called sine-Gordon model

$$S = \int d^2x \left[\frac{1}{2} \sum_{\mu=1}^{2} (\partial_\mu \phi)^2 - \Delta \cos(\beta\phi) \right]. \tag{3.1}$$

It appears in many problems in physics. Most notably it describes the Kosterlitz-Thouless transition in the 2 dimensional XY model (which in turn describes the superfluid transition in thin helium films, melting of two dimensional crystals etc).

1. Determine the values of β for which Δ is relevant or irrelevant, and β_c which makes it marginal. Plot the RG flow of Δ. You will need to use the results taken from Problem Set 4.

At those values where Δ is irrelevant it can be safely thrown out, which makes the sine-Gordon model trivial and automatically solves it. Where it is relevant the RG can no longer help us, and the sine-Gordon model in this regime can only be solved by a rather complicated technique called Bethe ansatz. The exception to this is a special point $\beta^2 = 4\pi$, called the *massive fermion* point.

2. Solve the sine-Gordon at $\beta^2 = 4\pi$ by fermionization (inverse to bosonization).