

Problem Set 6

Phys 7240
Due: Mar 30

1 Fractional Charge and Berry Phase

In this problem the Planck constant is set to one everywhere $\hbar = 1$. A charged particle moving in a magnetic field is described by the action

$$S = \int dt \left[\frac{m\dot{x}_\mu^2}{2} + \frac{e}{c} A_\mu \dot{x}_\mu \right]. \quad (1.1)$$

If a particle can be thought of as almost classical, its wave function acquires a phase. It can be calculating using the path integral approach.

$$\psi(x_1, t_1) = \int^{x(t_1)=x_1} \mathcal{D}x(t) e^{iS}. \quad (1.2)$$

A path integral for a quasiclassical particle is dominated by its classical trajectory $x(t)$, and the phase due to the magnetic field follows

$$\psi(x, t) = \exp \left[i \frac{e}{c} \int dt A_\mu \dot{x}_\mu \right] e^{iS_0} = \exp \left[i \frac{e}{c} \int dx_\mu A_\mu \right] e^{iS_0}, \quad (1.3)$$

where S_0 is the action computed along the trajectory $x(t)$ in the absence of the field.

1. Calculate this phase for a closed trajectory $x(t_1) = x(0)$ without self intersections in a two dimensional space with the magnetic field perpendicular to it. Show that it is equal to $\exp \left[i \frac{e}{c} \Phi \right]$ where Φ is the total flux of the magnetic field through the area enclosed by the trajectory $x(t)$.

The magnetic flux which corresponds to this phase equal to 2π is called the flux quantum, $\Phi_{FQ} = \frac{2\pi c}{e}$. However, the notion of a flux quantum depends on the charge of the particle moving in the magnetic field. For example, if an electron with charge e moves around an area with the total magnetic flux being $2\pi c/e$, it acquires a phase 2π . But if a different particle with charge q moves around the same area, it acquires the phase $2\pi q/e$.

Now suppose a Laughlin state is formed in a quantum Hall effect experiment at the filling fraction $\nu = 1/(2m + 1)$ and with a uniform magnetic field B . Suppose also that a Laughlin quasihole with charge $e_{QH} = e/(2m + 1)$ is transported around a trajectory which encloses one flux quantum for an electron. Then it should acquire a phase equal to

$$\varphi = 2\pi \frac{e_{QH}}{e} = \frac{2\pi}{2m + 1}. \quad (1.4)$$

This can be calculated as a Berry phase of the quasihole which is equal to

$$\varphi_{\text{Berry}} = i \oint d\eta \left\langle \psi_\eta \left| \frac{\partial \psi_\eta}{\partial \eta} \right. \right\rangle + i \oint d\eta^* \left\langle \psi_\eta \left| \frac{\partial \psi_\eta}{\partial \eta^*} \right. \right\rangle, \quad (1.5)$$

where

$$\psi_\eta(z_1, z_2, \dots) = C(\eta, \eta^*) \prod_i (\eta - z_i) \prod_{i < j} (z_i - z_j)^{2m+1} e^{-\sum_i \frac{z_i z_i^*}{4l^2}} \quad (1.6)$$

is the wave function of a Laughlin state with one quasihole at the position η , and $l = \sqrt{c/eB}$ is the magnetic length. Here $C(\eta, \eta^*)$ is yet unknown normalization factor. In the remaining part of this problem you will calculate the normalization factor and the phase.

2. Calculate $C(\eta, \eta^*)$ by demanding that the norm of the wave function Eq. (1.6) is proportional to unity. To do that, recast the normalization integral

$$\langle \psi_\eta | \psi_\eta \rangle = \int d^2 z_1 d^2 z_2 \dots \psi_\eta^* \psi_\eta = 1 \quad (1.7)$$

in the form of a partition function of a fictitious two dimensional plasma and think about how C must be chosen in order for the charged particle represented by η to interact properly with the charged background.

3. Show (using Eq. (1.5) and Eq. (1.6)) that the Berry phase can now be recast in the form

$$\varphi_{\text{Berry}} = -i \oint d\eta \left\langle \frac{\partial \psi_\eta}{\partial \eta} \left| \psi_\eta \right. \right\rangle + i \oint d\eta^* \left\langle \psi_\eta \left| \frac{\partial \psi_\eta}{\partial \eta^*} \right. \right\rangle = -i \oint d\eta \frac{\partial \log C^*}{\partial \eta} + i \oint d\eta^* \frac{\partial \log C}{\partial \eta^*} \quad (1.8)$$

4. Use the explicit form of $C(\eta, \eta^*)$ found in part 2 to calculate the Berry phase and show that it is equal to Eq. (1.4).

This provides another proof that the Laughlin quasiparticles carry fractional charge.