Problem Set 5

Phys 7240
Due: Mar 16

1 Spin and Charge Separation in a Luttinger Liquid

So far we discussed bosonization of spinless electrons. In real life, electrons carry spin and their Hamiltonian can be written as

$$H = \sum_p \frac{p^2}{2m} \left(a^\dagger_{p\uparrow} a_{p\uparrow} + a^\dagger_{p\downarrow} a_{p\downarrow} \right). \tag{1.1}$$

Upon bosonization, we find (each spin component can be bosonized independently)

$$H = \frac{1}{2\pi} \sum_{\sigma=\uparrow,\downarrow} \int dx \left[(\partial_x \theta_\sigma)^2 + (\partial_x \varphi_\sigma)^2 \right]. \tag{1.2}$$

The total charge density can be written as $j^0 = \partial_x \theta_\downarrow + \partial_x \theta_\uparrow$. Consequently, the bosonized fields can be split into "charge" and "spin" parts:

$$\theta_c = \theta_\uparrow + \theta_\downarrow, \;\; \theta_s = \theta_\uparrow - \theta_\downarrow, \tag{1.3}$$

and equivalently with φ. The Hamiltonian decouples into spin and charge Hamiltonians. This is called spin-charge separation in a Luttinger liquid.

The most general interaction between electrons can be written as

$$H_{\text{int}} = \sum_{\sigma_1 \sigma_2} \sum_{k_1, k_2, q} V_{\sigma_1 \sigma_2}(q, k_1, k_2)\, a^\dagger_{\sigma_1}(k_1) a^\dagger_{\sigma_2}(k_2) a_{\sigma_2}(k_2 + q) a_{\sigma_1}(k_1 - q). \tag{1.4}$$

1. Split this interaction into four distinct terms with k_1, k_2, $k_2 + q$ and $k_1 - q$ restricted to $\pm k_F$ and express them in terms of ψ_L, ψ_R.
2. Bosonize ψ_L, ψ_R and express the interaction in terms of the fields θ_s, θ_c, φ_s, φ_c.
3. You will find that some of the interaction terms can be reduced to those which simply change the parameters v and g of the charge and spin Hamiltonians, while others will be more complicated. Using only the interactions of the first type, find v_s, v_c, g_s, g_c in terms of the parameters of the interaction.

2 Kubo Formula and Luttinger Liquid Conductance

If one applies a uniform electric field E along a wire between the points $0 < x < L$, the field produces current. The current is given by $I = GEL = GU$, where the conductance G is given by

$$G = \frac{e^2}{\hbar} \lim_{\omega \to 0} \frac{1}{\omega L^2} \int_0^L dx \int_0^L dx' \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \left\langle J(x,\tau) J(x',0) \right\rangle. \tag{2.1}$$

Here τ is imaginary time, and $J = \partial_\tau \theta/\pi$ is the electric current.
1. Use this formula to evaluate conductance of Luttinger liquid.
2. Derive this formula.

3 2D Electrons in a Magnetic Field

Electrons moving in two dimensions in a constant magnetic field are described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left[\left(\partial_x + \frac{ie}{2\hbar c}By\right)^2 + \left(\partial_y - \frac{ie}{2\hbar c}Bx\right)^2\right] \tag{3.1}$$

The common notations are $\omega = eB/mc$, the Larmor frequency, and $l = \sqrt{\hbar c/eB}$, the magnetic length.
1. Show that these are mutually commuting creation and annihilation operators:

$$\begin{aligned} a &= \sqrt{2}l\bar{\partial} + \frac{1}{2\sqrt{2}l}z, \ a^\dagger = -\sqrt{2}l\partial + \frac{1}{2\sqrt{2}l}\bar{z}, \\ b^\dagger &= \sqrt{2}l\bar{\partial} - \frac{1}{2\sqrt{2}l}z, \ b = -\sqrt{2}l\partial - \frac{1}{2\sqrt{2}l}\bar{z}, \end{aligned} \tag{3.2}$$

where $z = x + iy$ and $\partial = \frac{\partial}{\partial z}$.
2. Show that the Hamiltonian can be written as

$$H = \frac{\hbar\omega}{2}\left[a^\dagger a + a a^\dagger\right]. \tag{3.3}$$

Find the wave functions in the lowest Landau level and find what happens if b and $b^\dagger$ are applied to them.
3. Electrons with spin moving in a magnetic field $B(x, y)$ which varies in space are described by

$$H = -\frac{\hbar^2}{2m}\left[\left(\partial_x + \frac{ie}{\hbar c}A_x(x,y)\right)^2 + \left(\partial_y + \frac{ie}{\hbar c}A_y(x,y)\right)^2\right] \pm \frac{e\hbar}{2mc}B(x,y), \tag{3.4}$$

where $\pm$ stands for spin up or spin down. Found the ground state of this Hamiltonian.